When Fourier Analysis Meets Ergodic Theory and Number Theory

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Birkhoff's ergodic theorem

Birkhoff's ergodic theorem establishes pointwise convergence for "time averages" to a "space averages" as follows:

Theorem (Birkhoff)

Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure preserving system, and $f \in L^1_\mu$. Then,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f \circ T^n(x) = \int_X f \, d\mu$$

μ-a.e. for $x \in X$.

The quantity on the left is the "time average," and the quantity on the right is the "space average."

Example

Let $(\mathbb{T}, \mathcal{A}, \mu)$ be the Lebesgue measure space on the torus $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$, α an irrational number, and $T_\alpha(x) = x + \alpha \pmod 1$. Let A be a measurable set, and consider the indicator function

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Birkoff's ergodic theorem with $f = \mathbb{1}_A$ gives that for $\mu-$almost every $x \in \mathbb{T}$,

$$\lim_{N\to\infty} \mathbb{P}(T_\alpha^n(x) \in A \mid 1 \leq n \leq N) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_A \circ T_\alpha^n(x) = \int_{\mathbb{T}} \mathbb{1}_A dx$$
$$= \mu(A).$$

Our goal

We aim to establish the following theorem:

Theorem

Let $(X, \mathcal{B}, \mu, T)$ be a σ-finite dynamical system with T invertible. Let $c > 1$ be sufficiently close to 1, and $(p_n)_{n\in\mathbb{N}}$ be the standard enumeration of primes. For all $r \in (1, \infty)$ and $f \in L^r_\mu$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f \circ T^{\lfloor p_n^c \rfloor}(x) \text{ exists for } \mu\text{-a.e. } x \in X.$$

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Using summation by parts, it can be seen that it suffices to show pointwise convergence for

$$\lim_{N \to \infty} \frac{1}{N} \sum_{p \leq N} (\log p)(f \circ T^{\lfloor p^c \rfloor}(x)).$$

Sketch of proof

1. By Calderon's transference principle, it suffices to consider the integer shift system $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}), \#, T)$ where $\#$ is the counting measure and $T(x) = x - 1$.
2. Use a result establishing pointwise convergence on L^2 given appropriate control for the oscillation seminorm.
3. The set $L^2 \cap L^p$ is dense in L^p, so appealing to maximal inequalities for $p > 1$ on $A_n(f)$, it is established for L^p for $p > 1$.
4. The oscillation and L^p maximal inequalities are attained by analyzing the corresponding "multiplier" on the frequency side, by using techniques from analytic number theory to bound exponential sums.

Averaging operators

Recall that it suffices to show pointwise convergence for

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In the case where $T(x) = x - 1$, $f \circ T^{\lfloor p^c \rfloor}(x) = f(x - \lfloor p^c \rfloor)$, and so we can consider the averaging operator

$$A_t f(x) = \frac{1}{t} \sum_{p\leq t} (\log p) f(x - \lfloor p^c \rfloor).$$

Another averaging operator we consider is

$$B_t f(x) = \frac{1}{t} \sum_{m\leq t^c} \frac{1}{c} m^{\frac{1}{c}-1} f(x-m).$$

Two multipliers

After taking a Fourier transform, the averages transform into multipliers on the frequency side. One of these is

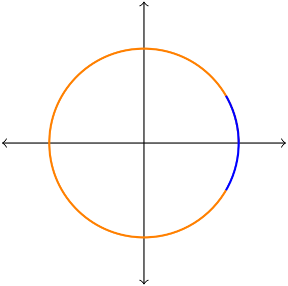
$$a_t(\xi) = \frac{1}{t} \sum_{p \le t} (\log p) e^{2\pi i \lfloor p^c \rfloor \xi},$$

which is normalized in the sense that $a_t(0) \to 1$ as $t \to \infty$. We can show a_t is "close" to the following trigonometric polynomial b_t in some sense:

$$b_t(\xi) = \frac{1}{t} \sum_{m \le t^c} \frac{1}{c} m^{\frac{1}{c}-1} e^{2\pi i m \xi}.$$

We study this closeness using ideas from the Hardy–Littlewood circle method.

The circle method



We split the circle into a major arc and minor arc, as represented in the diagram to the left. The blue arc is the major arc, which we shall denote by M_t, and the one in orange is the minor arc m_t. We study the difference $a_t - b_t$ on each arc separately, and want to obtain good estimates on the L^∞ norms $\|(a_t - b_t)\mathbb{1}_{M_t}\|_{L^\infty}$ and $\|(a_t - b_t)\mathbb{1}_{m_t}\|_{L^\infty}$.

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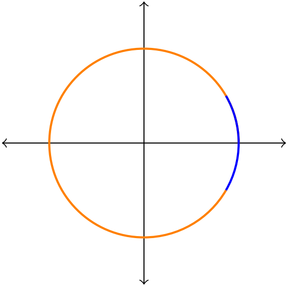

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The minor arc is where both a_t and b_t are small, and so their difference happens to be small as a consequence. The major arc is where their difference is small because the two are close to each other.

Analytic number theory techniques

Many of the things we have to estimate have sums involving primes. For example, in the process of obtaining minor arc estimates, we had to study the exponential sum

$$\sum_{n \leq N} \Lambda(n) e^{2\pi i x n^c},$$

where Λ is the von Mangoldt function. To estimate such sums, we made use of some standard tools from analytic number theory, such as summation by parts and Vaughan's identity.

Vaughan's identity

The definition of the von Mangoldt function is as follows:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with } p \text{ prime for some integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Vaughan's identity allows us to write $\Lambda(n)$ as three different sums involving the Möbius function μ:

$$\Lambda(n) = \sum_{\substack{b|n \\ b \leq y}} \mu(b) \log\left(\frac{n}{b}\right) - \sum\sum_{\substack{bd|n \\ b \leq y, d \leq z}} \mu(b)\Lambda(d) + \sum\sum_{\substack{bd|n \\ b > y, d > z}} \mu(b)\Lambda(d)$$

where $y, z \geq 1$ and $n > z$.

Oscillations

Definition

Let $a_n(x) : X \to \mathbb{C}$ be a sequence of measurable functions. We define the 2-**oscillation seminorm** of the family $(a_n(x) : n \in \mathbb{I} \subseteq \mathbb{N})$ at x by

$$O^2_{I,J}(a_n(x) : n \in \mathbb{I}) = \left(\sum_{j=0}^{J-1} \sup_{I_j \le t \le I_j+1} |a_t(x) - a_{I_j}(x)|^2 \right)^{\frac{1}{2}},$$

where $I = (I_j : 0 \le j \le J)$ is a strictly increasing sequence of length $J+1$ and takes values in $\mathbb{N}$. The set of all such sequences is denoted $\mathfrak{S}_J(\mathbb{N})$.

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Theorem

Let $a_n(x)$ be as above. If for $J \in \mathbb{N}$, $\sup_{I \in \mathfrak{S}_J(\mathbb{N})} \| O^2_{I,J}(a_n(x) : n \in \mathbb{I}) \|_{\ell^2}$ $= o(J^{\frac{1}{2}})$, then $\lim_{N \to \infty} a_n(x)$ exists for μ-a.e. x.

Maximal inequalities

Theorem

Let $f \in \ell^2$. *Then, for* $\lambda \in (1, \infty)$,

$$\sup_{J \in \mathbb{N}} \sup_{I \in \mathfrak{S}_J(\mathbb{N})} \| O^2_{I,J}(A_{\lfloor \lambda^n \rfloor}(x) : n \in \mathbb{I}) \|_{\ell^2} \leq C_\lambda \| f \|_{\ell^2}.$$

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Once oscillation inequalities are established in ℓ^2, we establish maximal inequalities on ℓ^p.

Theorem

Let $p \in (1, \infty)$, $f \in \ell^p$. Then,

$$\left\| \sup_{n \geq 1} A_n f(x) \right\|_{\ell^p} \leq C_p \|f\|_{\ell^p}.$$

Then, we transfer oscillations and maximal inequalities using Calderon's transference principle. Proceeding as in the proof of the Lebesgue differentiation theorem, we obtain the main result.

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References

- Leonidas Daskalakis. Roth's theorem and the Hardy–Littlewood majorant problem for thin subsets of primes, 2022.
- Manfred Einsiedler and Thomas Ward, *Ergodic Theory: with a view towards Number Theory*. Springer London, 2011.
- Loukas Grafakos, *Classical Fourier Analysis*. Springer New York, 2014.
- Henyrk Iwaniec and Emmanuel Kowalski. *Analytic Number Theory*. American Mathematical Society, 2004.
- M. B. S. Laporta and D. I. Tolev. On an equation with prime numbers. *Mathematical Notes*, 57(6):654–657, 6 1995.
- Mariusz Mirek, Tomasz Z. Szarek, and James Wright. Oscillation inequalities in ergodic theory and analysis: one-parameter and multi-parameter perspectives, 2022.
- D. I. Tolev. On a diophantine inequality involving prime numbers. *Acta Arithmetica*, 61(3):289–306, 1992.