Week 1 Measure Theory Exercises

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Exercise 1

Exercise: Let $\mathcal{A}$ be an algebra of subsets of X. Prove that $\mathcal{A}$ is a σ-algebra if and only if it is closed under increasing countable unions.

Solution:

Proof. By definition, if $\mathcal{A}$ is a σ-algebra, it is closed under countable unions, including increasing countable unions, so it remains to show the other direction. Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ be an arbitrary sequence of sets. Define

$$B_n = \bigcup_{m=1}^{n} A_n$$

Clearly $B_n \subseteq B_{n+1}$, so $\{B_n\}_{n\in\mathbb{N}}$ is a countable increasing sequence, and since $\mathcal{A}$ is an algebra and each B_n is formed by a finite union of members of $\mathcal{A}$, each B_n is in $\mathcal{A}$. We have

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \in \mathcal{A}.$$

□

Exercise 2

Exercise: Let X be a non-empty set and let m^* be an outer measure on X with $m^*(X) < \infty$. Let $\nu^*(E) = (m^*(E))^{1/2}$ for all $E \subseteq X$. Prove that ν^* is also an outer measure. Prove that a set E belongs to the Carathéodory σ-algebra of ν^* iff $\nu^*(E)\nu^*(X\backslash E) = 0$.

Solution: First, we show that ν^* is also an outer measure.

Proof. First, we see that $\nu^*(\emptyset) = (m^*(\emptyset))^{1/2} = 0^{1/2} = 0$. Next, for $A \subseteq B$, we know that $\mu^*(A) \leq \mu^*(B)$ by the properties of an outer measure. Since the square-root function is monotone increasing, this means $(\mu^*(A))^{1/2} \leq (\mu^*(B))^{1/2}$, so $\nu^*(A) \leq \nu^*(B)$. Finally, we use the subadditivity of the square root function to see that

$$\nu^*\left(\bigcup_{n=1}^{\infty} A_n \infty\right) = \left(m^*\left(\bigcup_{n=1}^{\infty} A_n\right)\right)^{1/2} \leq \left(\sum_{n=1}^{\infty} m^*(A_n)\right)^{1/2} \leq \sum_{n=1}^{\infty} (m^*(A_n))^{1/2} = \sum_{n=1}^{\infty} \nu^*(A_n)$$

□

Now, we show that E belongs to the Carathéodory σ-algebra of ν^*, $\mathcal{M}(\nu^*)$, if and only if either $\nu^*(E) = 0$ or $\nu^*(X\backslash E) = 0$. Of course, if $\nu^*(E)$ or $\nu^*(X\backslash E)$ are zero, then they belong to this σ-algebra, so it remains to show the other direction.

Proof. Suppose $E \in \mathcal{M}(\nu^*)$. Then, we must have $\nu^*(X) = \nu^*(X \cap E) + \nu^*(X\backslash E)$. Squaring both sides, we see that $m^*(X) = m^*(X \cap E) + m^*(X\backslash E) + 2\nu^*(X \cap E)\nu^*(X\backslash E) = m^*(X \cap E) + m^*(X\backslash E)$, and since $X \cap E = E$, this means either $\nu^*(E) = 0$ or $\nu^*(X\backslash E) = 0$. □

Exercise 3

Exercise: Assume $E \subseteq \mathbb{R}$ has Lebesgue outer measure 0, i.e. $\lambda_1^*(E) = 0$, where λ_1^* is the one-dimensional Lebesgue outer measure. Prove that $\mathbb{R}\backslash E$ is dense in $\mathbb{R}$. Also prove that for all $A \subseteq \mathbb{R}$, we have $\lambda_1^*(A \cup E) = \lambda_1^*(A) = \lambda_1^*(A\backslash E)$.

Solution: First, we show that $\mathbb{R}\backslash E$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}$ with $x < y$ (so $\lambda_1^*((x,y)) > 0$). We must have $(x,y) \cap (\mathbb{R}\backslash E) \neq \emptyset$. To see why, suppose not; then $(x,y) \subseteq E$, and by monotonicity $\lambda_1^*(E) \geq \lambda_1^*((x,y)) > 0$, contradicting $\lambda_1^*(E) = 0$. Since x and y were arbitrary and $(x,y) \cap (\mathbb{R}\backslash E) \neq \emptyset$, $\mathbb{R}\backslash E$ is dense in $\mathbb{R}$. □

Now, we show that for all $A \subseteq \mathbb{R}$ that $\lambda_1^*(A \cup E) = \lambda_1^*(A) = \lambda_1^*(A\backslash E)$.

Proof. First we show the equality $\lambda_1^*(A \cup E) = \lambda_1^*(A)$. By subadditivity, we have

$$\lambda_1^*(A \cup E) \leq \lambda_1^*(A) + \lambda_1^*(E) = \lambda_1(A)$$

and since $A \subseteq A \cup E$ we have by monotonicity that $\lambda_1^*(A) \leq \lambda_1^*(A \cup E)$, showing that $\lambda_1^*(A \cup E) = \lambda_1^*(A)$. To show the equality $\lambda_1^*(A) = \lambda_1^*(A\backslash E)$, we first note that since $A\backslash E \subseteq A$, $\lambda_1^*(A\backslash E) \leq \lambda_1^*(A)$ by monotonicity. We can also see that $(A\backslash E) \cup (E \cap A) = A$, so by subadditivity

$$\lambda_1^*(A) = \lambda_1^*((A\backslash E) \cup (E \cap A)) \leq \lambda_1^*(A\backslash E) + \lambda_1^*(E \cap A) = \lambda_1^*(A\backslash E)$$

(we know $\lambda_1^*(E \cap A) = 0$ because $E \cap A \subseteq E$). Therefore, $\lambda_1^*(A\backslash E) = \lambda_1^*(A)$. □

Exercise 4

Exercise: For all $\delta \in (0,1)$ we construct a set C_δ in a similar manner to the Cantor set. We start from $[0,1]$ and we remove from every remaining interval of the k-th step a middle interval of length $\delta 3^{-k}$. Prove that C_δ is compact and with no isolated points. Prove that it does not contain an interval and prove that $\lambda_1(C_\delta) = 1 - \delta > 0$.

Solution: Compactness is easy to show: we take the closed interval $[0,1]$ and take away countably many open intervals (which is the same as intersecting with the complement of the interval, which is closed). As C_δ is a countable intersection of closed sets, and it is bounded, it must be compact. To show that $\lambda_1(C_\delta) = 0$, we consider that at the k-step, there are 2^{k-1} intervals remaining, and so we obtain

$$\begin{aligned}\lambda_1(C_\delta) &= 1 - \sum_{k=1}^{\infty} 2^{k-1}\left(\delta 3^{-k}\right) 1 - \frac{\delta}{2}\sum_{k=1}^{\infty} 2^k 3^{-k} = 1 - \frac{\delta}{2}\sum_{k=1}^{\infty}\left(\frac{2}{3}\right)^k = 1 - \frac{\delta}{2}\left(\frac{2/3}{1/3}\right) = 1 - \frac{\delta}{2}(2) \\ &= 1 - \delta.\end{aligned}$$

Now, we show that C_δ does not contain an interval. After the k-th step, notice that each of the remaining 2^k intervals have an equal length, which is

$$\frac{1}{2^k}\left(1 - \frac{\delta}{2}\sum_{k'=1}^{k}\left(\frac{2}{3}\right)^{k'}\right) = \frac{1-\delta}{2^k} + \frac{\delta}{3^k} \xrightarrow[k\to\infty]{} 0$$

Since the length of all remaining intervals tends to zero as $k \to \infty$, C_δ cannot contain any intervals. A similar argument shows that C_δ does not contain any isolated points: let $x \in C_\delta$ and $\epsilon > 0$. Then, since the length of each interval remaining after step k tends towards zero as $k \to \infty$, there exists a step where the length of each remaining interval has length $< \epsilon$. x must be contained in some interval, and is at most ϵ away from the endpoints of this interval (which are also in C_δ), so x is not an isolated point of C_δ.

Exercise 5

Exercise: Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Prove that f is integrable iff

$$\sum_{k=-\infty}^{\infty} 2^k m(\{x \in \mathbb{R} : |f(x)| > 2^k\}) < \infty$$

Solution: We will write $A_k = \{x \in \mathbb{R} : |f(x)| > 2^k\}$, and $B_k = \{x \in \mathbb{R} : 2^k < |f(x)| \leq 2^{k+1}\}$. Observe that the family of all B_k is pairwise disjoint, and that $A_k = \bigcup_{\ell=k}^{\infty} B_k$. As a result, $m(A_k) = \sum_{\ell=k}^{\infty} m(B_k)$. Moreover, notice that $A = \bigcup_{k=-\infty}^{\infty} A_k = \{x \in \mathbb{R} : f(x) \neq 0\}$, so $\int_{\mathbb{R}} f \, dm = \int_A f \, dm$.

Proof. ($\Rightarrow$) Assume f is integrable, so $\int_{\mathbb{R}} |f| \, dm < \infty$. We can see that

$$\sum_{k=-\infty}^{\infty} 2^k \mathbb{1}_{B_k} \leq |f|,$$

so

$$\int_{\mathbb{R}} |f| \, dm = \sum_{k=-\infty}^{\infty} \int_{B_k} |f| \, dm \geq \sum_{k=-\infty}^{\infty} \int_{B_k} 2^k \, dx = \sum_{k=-\infty}^{\infty} 2^k m(B_k)$$

(showing the rightmost sum is finite). Notice that all terms are positive, which allows us to use Fubini's theorem for series as follows:

$$\sum_{k=-\infty}^{\infty} 2^k m(A_k) = \sum_{k=-\infty}^{\infty} \sum_{\ell=k}^{\infty} 2^k m(B_\ell) \underbrace{=}_{\text{Fubini}} \sum_{\ell=-\infty}^{\infty} \sum_{k=\ell}^{\infty} 2^k m(B_\ell) = \sum_{\ell=-\infty}^{\infty} 2^{1+\ell} m(B_\ell) < \infty.$$

($\Leftarrow$) Assume $\sum_{k=-\infty}^{\infty} 2^k m(A_k) < \infty$. Since $B_k \subseteq A_k$, $\mathbb{1}_{B_k} \leq \mathbb{1}_{A_k}$, and thus

$$|f| \leq \sum_{k=-\infty}^{\infty} 2^{k+1} \mathbb{1}_{B_k} \leq \sum_{k=-\infty}^{\infty} 2^{k+1} \mathbb{1}_{A_k}.$$

Integrating yields

$$\int_{\mathbb{R}} |f| \, dm \leq \sum_{k=-\infty}^{\infty} 2^{k+1} m(A_k) < \infty$$

as desired. □

Exercise 6

Exercise: If f is Lebesgue integrable on $[-1, 1]$, prove that $\lim_{n\to\infty} \int_{-1}^{1} x^n f(x) \, dx = 0$.

Solution:

Proof. We shall use the dominated convergence theorem. Define $f_n(x) = x^n f(x)$. As f is Lebesgue integrable, it must be finite almost everywhere. Thus, we can see that

$$\lim_{n\to\infty} f_n(x) = 0 \text{ a.e.}$$

(wherever f is finite-valued). Moreover, on $[-1, 1]$, we know $|x^n| \leq 1$, so $|f_n(x)| = |x^n f(x)| \leq |f(x)|$. By the dominated convergence theorem, and using the fact that this pointwise limit is 0 a.e., we can see that

$$\lim_{n\to\infty} \int_{-1}^{1} x^n f(x) \, dx = \int_{-1}^{1} \lim_{n\to\infty} x^n f(x) \, dx = \int_{-1}^{1} 0 \, dx = 0.$$

□