Hardness of Approximation of the Steiner Tree Problem

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What is the Steiner Tree Problem?

Definition (The Steiner Tree Problem)

Given U, a subset of n points from a metric space X, what is the minimum cost of a spanning tree of $U \cup S$ over all $S \subset X$?

- The points in U are *terminals*, the points in S are *Steiner points*, and the spanning tree is a *Steiner tree*.

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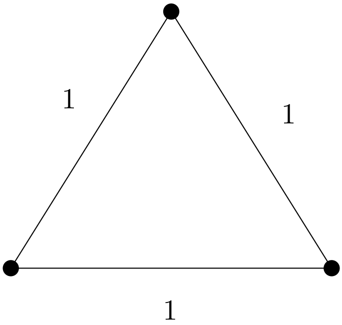

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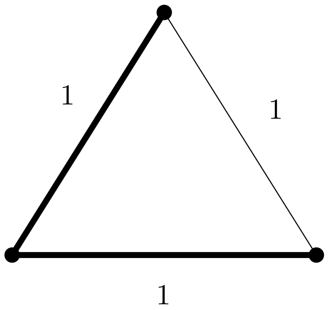

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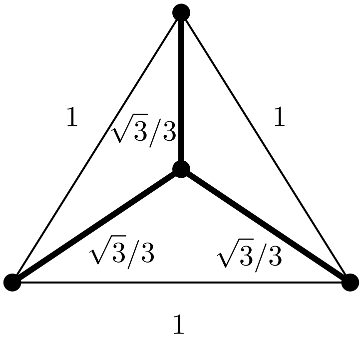

The Hamming Steiner Tree Problem

Definition (Hamming Min ST)

Given (U, k), with U subset of n points in n-dimensional Hamming space, is there is a Steiner tree of cost at most k?

- Hamming space is $\{0,1\}^n$ under the Hamming metric, where $\|x - y\|_0 = \sum_i x_i \oplus y_i$ (edit distance).

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- APX-hard means there is some ρ such that it is NP-hard to approximate solutions within $(1+\rho)$. But what is ρ?

A Diversion: Vertex Cover

Definition (MIN VC)

Given (G, k) with G a graph, does G have *vertex cover* of size k? A vertex cover is a subset of vertices containing an endpoint of each edge.

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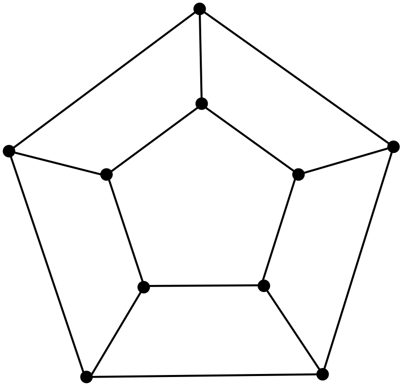

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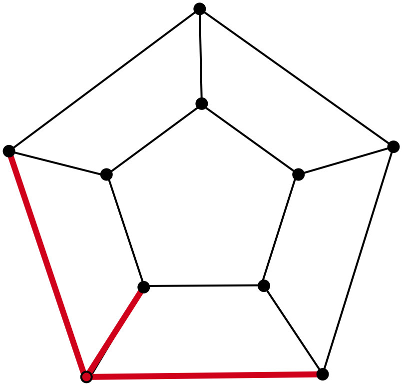

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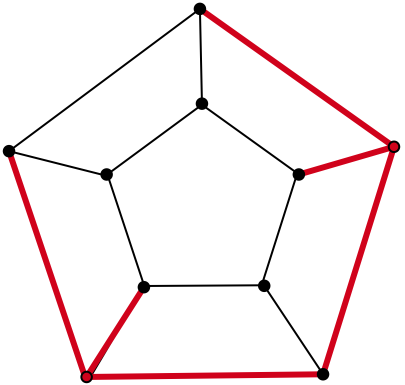

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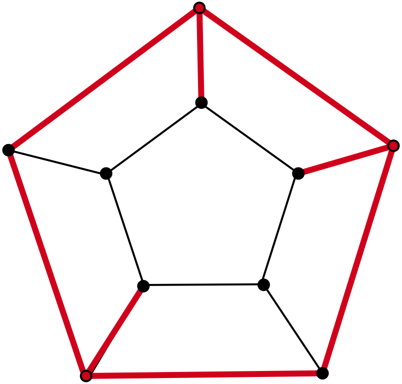

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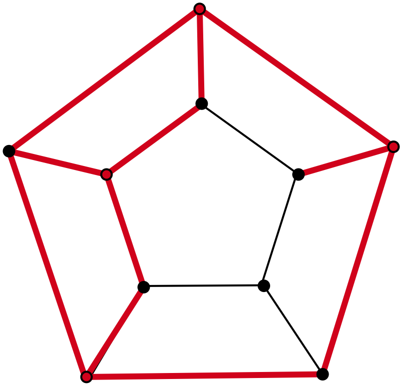

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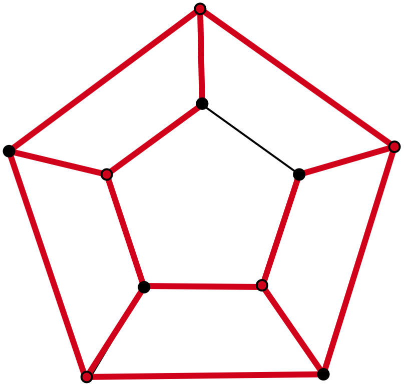

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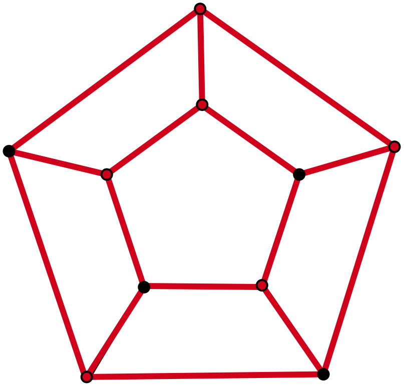

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- Why Min VC?

Theorem (Chlebík-Chlebíková 2008)

Min VC *is* NP*-hard to approximate within a factor of* $48/47$ *on* 4*-regular graphs.*

A Reduction From Vertex Cover

- Let $G = (\{v_1, v_2, \ldots, v_n\}, E)$ be a graph of size m.

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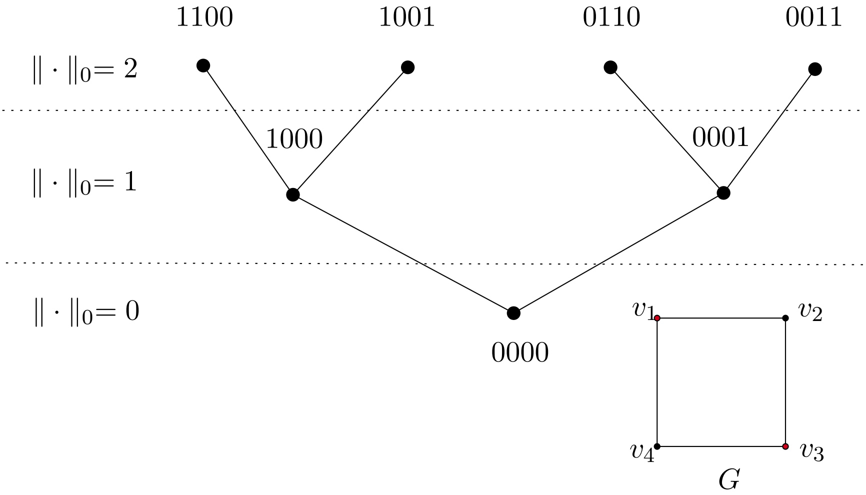
- Suppose G has a k-vertex cover.
- Then, $\mathcal{P}(G)$ has a Steiner tree of cost $m + k$.

Finding a Steiner tree of cost $m + k$



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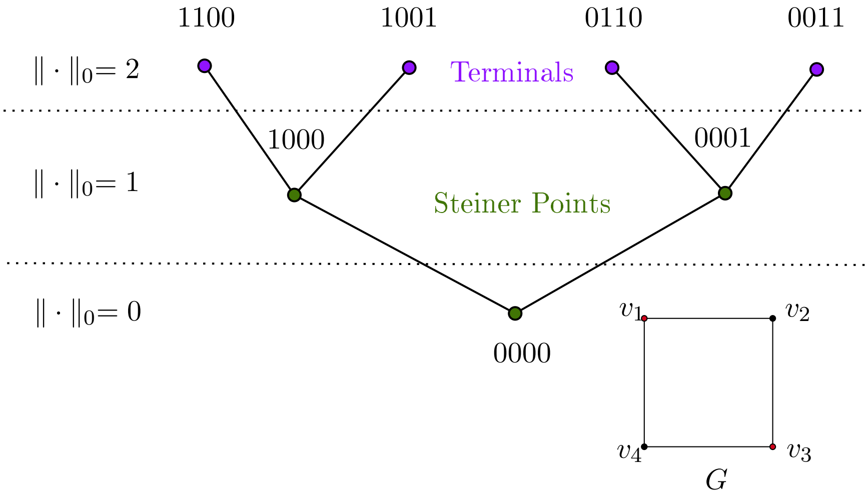

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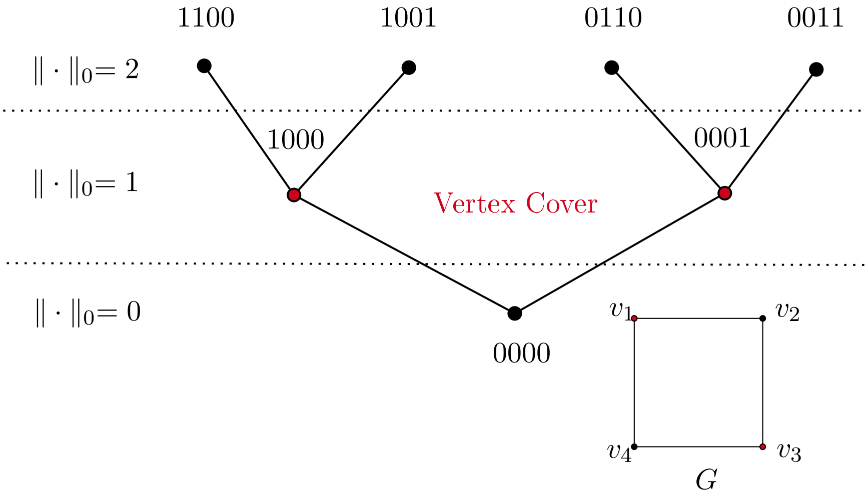

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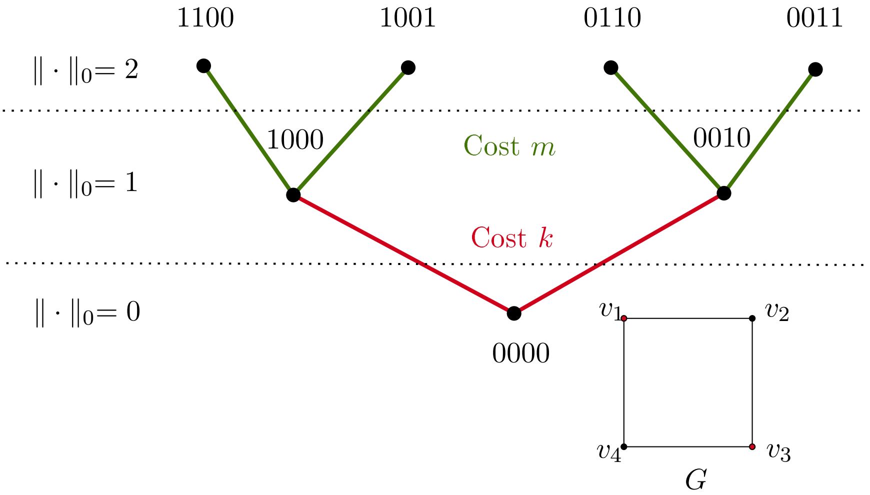

Our Results

Theorem (Embedding Vertex Cover in Hamming Steiner Trees)

For a 4-regular graph G of order n, $\mathcal{P}(G)$ has the following properties.

1. *If G has a vertex cover of size k, then $\mathcal{P}(G)$ has a Steiner tree T with $\mathsf{cost}(T) \leq 2n + k$.*
2. *If $\mathcal{P}(G)$ has a Steiner tree T with $\mathsf{cost}(T) \leq 2n + k$, then G has a vertex cover of size at most $k + 1$.*

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Theorem (Hamming Steiner Tree Hardness)

Hamming Min ST *is* NP*-hard to approximate within a factor of* 1.004.

Acknowledgements

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