Invariants and varieties of multidimensional matrices

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2014 June 29

We seek to adapt or generalize our methods for computing Grothendieck classes of matrix varieties to varieties of multidimensional matrices. To do so, we seek to understand some of the algebraic structure of multidimensional matrices. We adopt a linear algebra approach, treating multidimensional matrices as tensor products, and attempt to obtain algebraic invariants for them.

These notes were written during the course of an REU project at Rutgers University conducted by Anders Buch, Elliot Glazer, and the author.

1 Multidimensional matrices

We work over the field of complex numbers. Our first definition of a multidimensional matrix is a purely formal extension of the usual notion of a matrix. Some of the notation and terminology is now standard in the literature (originating in Gel’fand, Kapranov, and Zelevinsky’s work on discriminants), but some I have made up in the absence of other sources.

**Definition.** A *multidimensional matrix* is an array $A = (a_{i_0i_1...i_p})$, where each $a_{k_0k_1...k_p} \in \mathbb{C}$ and the indices $k_j$ run as

- $1 \leq i_0 \leq k_0$,
- $1 \leq i_1 \leq k_2$,
- $1 \leq i_2 \leq k_3$,
- $\vdots$
- $1 \leq i_k \leq k_p$.

We say that $A$ is of format $k_0 \times k_1 \times \cdots \times k_p$.

From here on, if we refer to “matrix” without the prefix “multidimensional” and the format is not specified, we refer to two-dimensional matrices as we would see in classical linear algebra.

This establishes multidimensional matrices as combinatorial objects. However, a more algebraic definition might be useful to examine their symmetry and invariance properties. In this spirit, let $V_0, \ldots, V_p$ be vector spaces of dimension $k_i$, $i = 0, \ldots, p$, and consider the tensor product $V_0 \otimes \cdots \otimes V_p$. This is a complex vector space of dimension $\prod_{i=0}^{p} k_j$, so it is isomorphic to the space of multidimensional matrices of format $k_0 \times \cdots \times k_p$. Therefore, once we fix a basis on each $V_i$, we may regard multidimensional matrices of format $k_0 \times \cdots \times k_p$ as elements of the tensor product.

In both approaches, we wish to extend the notion of elementary row and column operations from standard linear algebra.
Definition. Let $A$ be a multidimensional matrix of format $k_0 \times \cdots \times k_j \times \cdots \times k_p$. A slice of $A$ is a submatrix of format $k_0 \times \cdots \times \hat{k}_j \times \cdots \times k_p$ obtained from $A$ by holding one of the indices $i_j$ constant. Two slices are parallel if they are obtained by holding the same indices $i_j, i'_j$ constant.

There are $p + 1$ ways to cut a matrix of format $k_0 \times \cdots \times k_p$ into parallel slices.

Definition. A slice operation on a multidimensional matrix $A$ consists of one of the following:

1. interchanging any two parallel slices;
2. multiplying a slice by a nonzero scalar;
3. adding a scalar multiple of one slice to a parallel slice.

Slice operations are quite natural generalizations of row and column operations to matrices of larger format, we can generalize further:

Definition. Let $A$ be a multidimensional matrix of format $k_0 \times \cdots \times k_p$. A hyperslice of a multidimensional matrix $A$ of type $(i_1, \ldots, i_j)$, where the $i_j$ are increasing and $0 \leq i_j \leq p$, is a submatrix of format $k_0 \times \cdots \times \hat{i}_j \times \cdots \times k_p$ obtained from $A$ by holding the indices $i_j$ constant. Parallel hyperslices and hyperslice operations are defined analogously.

2 A linear algebra approach to slice operations

Our definition of multidimensional matrices as elements of a tensor product allow us to approach them from a linear algebra perspective. More precisely, let $\mathbb{C}^{(k_0, \ldots, k_p)}$ denote the complex vector space of multidimensional matrices of format $k_0 \times \cdots \times k_p$, and let $V_0, \ldots, V_p$ be vector spaces of dimensions $k_0, \ldots, k_p$, $k_j > 0$. Fixing a basis on each $V_j$, identify $V_0 \otimes \cdots \otimes V_p$ with the space of multidimensional matrices of format $k_0 \times \cdots \times k_p$. Analogously to a similar statement we made in a previous set of notes, we have the following:

Proposition 1. For all $0 \leq i \leq p - 1$,

$$V_0 \otimes \cdots \otimes V_i \otimes V_{i+1} \otimes \cdots \otimes V_p \cong \text{Hom}(V_0 \otimes \cdots \otimes V_i, V_{i+1} \otimes \cdots \otimes V_p).$$

Corollary 2. $\mathbb{C}^{(k_0, \ldots, k_p)} \cong \text{Hom}(V_0 \otimes \cdots \otimes V_i, V_{i+1} \otimes \cdots \otimes V_p)$.

The following is now well-defined:

Definition. Let $A \in V_0 \otimes \cdots \otimes V_p$. For all $0 \leq i \leq p - 1$, rank$_i(A)$ is equal to the rank of $A$ as an element of $\text{Hom}(V_0 \otimes \cdots \otimes V_i, V_{i+1} \otimes \cdots \otimes V_p)$.

Since bases are fixed on all $V_j$, this fixes bases on $V_0 \otimes \cdots \otimes V_i$ and $V_{i+1} \otimes \cdots \otimes V_p$ as well, and thus we identify $\text{Hom}(V_0 \otimes \cdots \otimes V_i, V_{i+1} \otimes \cdots \otimes V_p)$ with the set of $(\prod_{k=1}^i k_j) \times (\prod_{j'=i+1}^p k_j')$ complex matrices. Thus computing rank$_i(A)$ reduces to a standard linear algebra computation.

It seems natural to think that the collection of all integers rank$_i(A)$ would be some invariant for the multidimensional matrix $A$. In this spirit, we make the following definition:
**Definition.** The sequence of integers \((\text{rank}_0(A), \text{rank}_1(A), \ldots, \text{rank}_{p-1}(A))\) is called the *multirank* of \(A\), and is denoted \(\text{Rank}(A)\).

Ideally, multirank would naturally carry many of the invariance properties of the classical rank of a matrix. In light of this, we make the following conjecture:

**Conjecture 3.** \(\text{Rank}(A)\) is invariant under hyperslice operations.

**Example 1.** Let \(A = (a_{ij}), 1 \leq i \leq m, 1 \leq j \leq n\) be a matrix of format \(m \times n\), considered as an element of \(\mathbb{C}^m \otimes \mathbb{C}^n\). \(A = (a_{ij})\) is another way of writing

\[
A = \sum_{ij} a_{ij} e_i \otimes e_j = \sum_j \left( \sum_i a_{ij} e_i \right) \otimes e_j.
\]

Therefore the \(j\)-th column of \(A\) is given by the coefficients \(a_{ij}\), where \(j\) is fixed and \(i\) runs from 1 to \(m\). This example will be useful as a guide when we generalize to higher formats.

**Example 2.** Let \(A = (a_{ijk}), 0 \leq i, j, k \leq 1\) be a matrix of format \(2 \times 2 \times 2\), and consider \(A\) as an element of \(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2\). We will express \(A\) as an element of \(\text{Hom}(\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2)\) and of \(\text{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes \mathbb{C}^2)\).

\(A = (a_{ijk})\) is another way of writing

\[
A = \sum_{ijk} a_{ijk} e_i \otimes e_j \otimes e_k = \sum_k \left( \sum_{ij} a_{ijk} e_i \otimes e_j \right) \otimes e_k = \sum_{jk} \left( \sum_i a_{ijk} e_i \right) \otimes (e_j \otimes e_k),
\]

where \(e_i\) denotes the \(i\)-th standard basis vector in \(\mathbb{C}^2\).

The above equalities suggest how the matrix of \(A\) should look like as an element of \(\text{Hom}(\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2)\) and of \(\text{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes \mathbb{C}^2)\). For example,

\[
A = \sum_k \left( \sum_{ij} a_{ijk} e_i \otimes e_j \right) \otimes e_k
\]

tells us that as an element of \(\text{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes \mathbb{C}^2)\) (i.e. as a \(4 \times 2\) matrix) \(A\) has two columns (for \(k = 0, 1\)), whose coefficients are given by \(a_{ijk}\) holding the \(k\) fixed. That is,

\[
A = \begin{bmatrix}
a_{000} & a_{001} \\
a_{010} & a_{011} \\
a_{100} & a_{101} \\
a_{110} & a_{111}
\end{bmatrix} \in \text{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes \mathbb{C}^2).
\]

Similarly,

\[
A = \begin{bmatrix}
a_{000} & a_{001} & a_{010} & a_{011} \\
a_{100} & a_{101} & a_{110} & a_{111}
\end{bmatrix} \in \text{Hom}(\mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2).
\]

Let us see how slice operations change these matrices. Consider the matrix

\[
A = \begin{bmatrix}
a_{000} & a_{001} \\
a_{010} & a_{011} \\
a_{100} & a_{101} \\
a_{110} & a_{111}
\end{bmatrix} \in \text{Hom}(\mathbb{C}^2, \mathbb{C}^2 \otimes \mathbb{C}^2).
\]
As one can check, a slice of $A$ as a $2 \times 2 \times 2$ matrix always corresponds to either one of the columns of $A$ as a $2 \times 4$ matrix, or as a $2 \times 2$ submatrix obtained by deleting two rows. (When doing this, it is convenient to think of $a_{ijk}$ as labeling the vertices of the cube $[0, 1]^3$.) This tells us that:

1. multiplying a slice by a scalar is the same as multiplying either a column or two rows by a scalar;
2. switching two slices is the same as either swapping the columns or switching two pairs of rows;
3. adding a multiple of a slice to another is the same as either adding a multiple of a column to another, or adding a multiple of a pair of rows to the remaining pair of rows.

Therefore this tells us that $\text{rank}_0(A)$ is a slice operation invariant. We can do the same in the $4 \times 2$ matrix representation to show that $\text{rank}_1(A)$ is a slice operation invariant. Therefore $\text{rank}(A)$ is a slice operation invariant in the $2 \times 2 \times 2$ format case.

However, Example 2 also shows quite clearly that $\text{rank}(A)$ is not a hyperslice invariant. For example, the hyperslice consisting of the coefficients $a_{10\ast}$, where $\ast$ is indeterminate, is not a collection of rows or columns in the $2 \times 4$ matrix shown above, so an operation involving that hyperslice is not rank-preserving for that matrix. In light of this, we modify Conjecture 3:

**Conjecture 4.** $\text{rank}(A)$ is a slice operation invariant.

This, in fact, is quite easy:

**Proof.** To specify a slice, we must fix one of the indices in $a_{j_0 \cdots j_p}$. Consider a representation of $A$ as an element of $\text{Hom}(V_0 \otimes \cdots \otimes V_i, V_{i+1} \otimes \cdots \otimes V_p)$: for example, with $\dim(V_r) = 2$ for all $r$ and $p = 4$, $i = 2$, we have

\[
\begin{array}{c|cccc}
& 00 & 01 & 10 & 11 \\
00 & a_{0000} & a_{0001} & a_{0010} & a_{0011} \\
01 & a_{0100} & a_{0101} & a_{0110} & a_{0111} \\
10 & a_{1000} & a_{1001} & a_{1010} & a_{1011} \\
11 & a_{1100} & a_{1101} & a_{1110} & a_{1111} \\
\end{array}
\]

where the leftmost column represents the $ij$-part of the index $a_{ijkl}$ and the topmost column represents the $k\ell$-part. Fixing exactly one index in $a_{ijkl}$ corresponds to selecting either a collection of columns (and only columns) in this representation, or a collection of rows (and only rows) in this representation. For example, fixing $i = 0$ corresponds to selecting the first and second rows of the sample representation above. From here it is quite obvious that a slice operation involving a collection of rows is really a composition of several elementary row operations, and a slice operation involving a collection of columns is a composition of elementary column operations. Row and column operations do not affect $\text{rank}_r(A)$, so $\text{rank}(A)$ is invariant under slice operations. \[\square\]

In particular, suppose the left column consisted of just one index, which would correspond to $A$ as an element of $\text{Hom}(V_0, V_1 \otimes \cdots \otimes V_p)$; in the above example, we would obtain

\[
\begin{array}{c|cccccccc}
& 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\
00 & a_{0000} & a_{0001} & a_{0010} & a_{0011} & a_{0100} & a_{0101} & a_{0110} & a_{0111} \\
10 & a_{1000} & a_{1001} & a_{1010} & a_{1011} & a_{1100} & a_{1101} & a_{1110} & a_{1111} \\
\end{array}
\]
Then a slice fixing the first index $i$ of $a_{ijkl}$ is simply a row of the resulting $(k_0) \times (\prod_{r}^{p} k_r)$ matrix, so slice operations for this index are precisely row operations. Therefore we can put this matrix into reduced row echelon form by slice operations, and rank$_0(A)$ is precisely the number of nonzero rows that remain after putting this matrix into reduced row echelon form. Similarly, rank$_{p-1}(A)$ is the number of nonzero columns that remain after putting the corresponding $(\prod_{r}^{p-1} k_r) \times k_p$ matrix into reduced column echelon form.

It is not necessarily true that rank$_0(A) = $ rank$_{p-1}(A)$: as an example,

\[
\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

is a $2 \times 4$ matrix of rank 2, so rank$_0(A) = 2$. However, the matrix as a $4 \times 2$ matrix is

\[
\begin{array}{cccccccccc}
0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

which is rank 1, so rank$_3(A) = 1$.

### 3 Slice matrices

The above choices of matrices were rather convenient for working with slice operations, and there is no reason not to consider fixing indices other than the first and last.

**Definition.** Let $A = (a_{i_0 \cdots i_p})$ be a matrix of format $k_0 \times \cdots \times k_p$. For $0 \leq j \leq p$, the slice matrix slice$_j(A)$ is the $k_j \times (\prod_{\ell \neq j} k_\ell)$ matrix whose $i_j$-th row $(1 \leq i_j \leq k_j)$ consists of the coefficients $a_{i_0 \cdots i_j \cdots i_p}$.

**Example 3.** For $A = (a_{i_0i_1i_2i_3})$, there are 4 slice matrices:

\[
\begin{array}{cccccccccc}
\text{slice}_0(A) = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{0000} & a_{0001} & a_{0010} & a_{0011} & a_{0100} & a_{0101} & a_{1010} & a_{1011} & a_{0111} \\
1 & a_{1000} & a_{1001} & a_{1010} & a_{1011} & a_{1100} & a_{1101} & a_{1110} & a_{1111} & a_{1111} \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
\text{slice}_1(A) = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{0000} & a_{0001} & a_{0010} & a_{0011} & a_{1000} & a_{1001} & a_{1010} & a_{1011} & a_{1101} \\
1 & a_{0100} & a_{0101} & a_{0110} & a_{0111} & a_{1100} & a_{1101} & a_{1110} & a_{1111} & a_{1111} \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
\text{slice}_2(A) = & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{0000} & a_{0001} & a_{0010} & a_{0011} & a_{1000} & a_{1001} & a_{1010} & a_{1011} & a_{1101} \\
1 & a_{0100} & a_{0101} & a_{0110} & a_{0111} & a_{1100} & a_{1101} & a_{1110} & a_{1111} & a_{1111} \\
\end{array}
\]

\[
\begin{array}{cccccccccc}
\end{array}
\]

5
\[
\text{slice}_3(A) = \begin{bmatrix}
0 & 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\
1 & a_{0000} & a_{0010} & a_{0100} & a_{0110} & a_{1000} & a_{1010} & a_{1100} & a_{1110} \\
a_{0001} & a_{0011} & a_{0101} & a_{0111} & a_{1001} & a_{1011} & a_{1101} & a_{1111}
\end{bmatrix}
\]

The coefficients of each slice \( j(A) \) are obtained by putting the indices in the leftmost column in the \( j \)-th place.

Every row of slice \( j(A) \) is a parallel slice of \( A \), and over all \( j \) the collection of rows forms the set of all slices of \( A \). All slice operations can thus be described as row operations on these slice matrices. We could also have defined these as column matrices and slice operations would then correspond to column operations. We note that when \( A \) is of format \( k_0 \times k_1 \), i.e. a classical matrix, \( \text{slice}_0(A) \) is just \( A \) itself and \( \text{slice}_1(A) \) is the transpose of \( A \).

Essentially the same proof as in Conjecture 4 shows that \( \text{rank}(\text{slice}_j(A)) \) is a slice operation invariant for all \( j \). This provides another invariant, similar to \( \text{Rank}(A) \):

**Definition.** The slice rank \( \text{slr}(A) \) of \( A \) is the integer vector \( \text{slr}(A) = (\text{rank}(\text{slice}_0(A)), \ldots, \text{rank}(\text{slice}_p(A))) \).

### 4 A naïve approach to Grothendieck classes

Here is one way we might try to compute Grothendieck classes for varieties of these matrices.

**Example 4.** We will work with the space of \( 2 \times 2 \times 2 \) matrices, identified with the tensor product of two-dimensional representations \( U \otimes V^* \otimes W \), where

\[
U = \mathbb{C}[u_0] \oplus \mathbb{C}[u_1], \\
V = \mathbb{C}[v_0] \oplus \mathbb{C}[v_1], \\
W = \mathbb{C}[w_0] \oplus \mathbb{C}[w_1].
\]

There are actually many ways to take the tensor product here; \( U^* \otimes V^* \otimes W \) might be another example. We chose \( U \otimes V^* \otimes W \) because it plays nicely with the Hom functor:

\[
\text{Hom}(\text{Hom}(U,V),W) = \text{Hom}(U^* \otimes V,W) = (U^* \otimes V)^* \otimes W = U \otimes V^* \otimes W.
\]

If we apply \( \text{Hom}(-,Y) \) (\( Y \) another representation) repeatedly on the left, we get a tensor product that formally alternates between vector spaces and duals, which may be something we want. For now let’s just proceed formally.

\( U \otimes V^* \otimes W \) is then the representation

\[
\bigoplus_{0 \leq i,j,k \leq 1} \mathbb{C}[u_i - v_j + w_k],
\]

and the coordinate ring of \( U^* \otimes V^* \otimes W \) is \( \mathbb{C}[x_{ijk}] \), where \( 0 \leq i,j,k \leq 1 \), is given a grading corresponding to: \( \text{deg}(x_{ijk}) = u_i - v_j + w_k \).

Suppose we want to consider the collection of all \( 2 \times 2 \times 2 \) matrices \( A \) with slice rank condition

\[
\text{slr}(A) = (1,1,1),
\]
i.e. \( \text{rank}(\text{slice}_\ell(A)) = 1 \) for \( \ell = 0, 1, 2 \). This means that the \( 2 \times 2 \) minors of each slice matrix must all vanish, giving us a system of polynomial equations that defines this collection. The slice matrices are

\[
\begin{align*}
\text{slice}_0(A) &= \begin{pmatrix} 0 & a_{000} & a_{001} & a_{010} & a_{011} \\ 1 & a_{100} & a_{101} & a_{110} & a_{111} \end{pmatrix} \\
\text{slice}_1(A) &= \begin{pmatrix} 0 & a_{000} & a_{010} & a_{101} \\ 1 & a_{010} & a_{011} & a_{110} \end{pmatrix} \\
\text{slice}_2(A) &= \begin{pmatrix} 0 & a_{000} & a_{010} & a_{101} \\ 1 & a_{010} & a_{011} & a_{111} \end{pmatrix}
\end{align*}
\]

The collection of distinct \( 2 \times 2 \) minors is the following 12 (homogeneous!) polynomials with the following degrees: (these should be double-checked because there’s about 7 in 10 odds I’ve made a typo somewhere)

\[
\begin{align*}
p(a_{ijk}) & \quad \text{deg}(p) \\
a_{000}a_{101} - a_{001}a_{100} & \quad u_0 + u_1 - 2v_0 + w_0 + w_1 \\
a_{000}a_{110} - a_{010}a_{100} & \quad u_0 + u_1 - v_0 - v_1 + 2w_0 \\
a_{000}a_{111} - a_{011}a_{100} & \quad u_0 + u_1 - v_0 - v_1 + w_0 + w_1 \\
a_{001}a_{110} - a_{010}a_{101} & \quad u_0 + u_1 - v_0 - v_1 + w_0 + w_1 \\
a_{001}a_{111} - a_{011}a_{101} & \quad u_0 + u_1 - v_0 - v_1 + 2w_1 \\
a_{010}a_{111} - a_{011}a_{110} & \quad u_0 + u_1 - 2v_1 + w_0 + w_1 \\
a_{000}a_{011} - a_{001}a_{010} & \quad 2u_0 - v_0 - v_1 + w_0 + w_1 \\
a_{000}a_{111} - a_{101}a_{100} & \quad u_0 + u_1 - v_0 - v_1 + w_0 + w_1 \\
a_{001}a_{110} - a_{100}a_{111} & \quad u_0 + u_1 - v_0 - v_1 + w_0 + w_1 \\
a_{100}a_{111} - a_{101}a_{110} & \quad 2u_1 - v_0 - v_1 + w_0 + w_1 \\
a_{000}a_{111} - a_{110}a_{001} & \quad u_0 + u_1 - v_0 - v_1 + w_0 + w_1 \\
a_{010}a_{101} - a_{100}a_{011} & \quad u_0 + u_1 - v_0 - v_1 + w_0 + w_1
\end{align*}
\]

The corresponding system of polynomial equations (which has some redundancies) determines an affine algebraic set, and since the polynomials are homogeneous also a projective algebraic set. In principle we now have all of the information required to compute the Grothendieck class of this set by taking a graded resolution.

**Example 5.** Here is what may be an interesting variety to study. For a \( 2 \times 2 \times 2 \) matrix \( A = (a_{ijk}) \) as above, Cayley’s second hyperdeterminant is defined by

\[
\text{Det}(A) = a^2_{000}a^2_{111} + a^2_{001}a^2_{110} + a^2_{010}a^2_{101} + a^2_{100}a^2_{011} \\
- 2a_{000}a_{001}a_{110}a_{111} - 2a_{000}a_{010}a_{101}a_{111} - 2a_{000}a_{011}a_{100}a_{111} \\
- 2a_{001}a_{010}a_{101}a_{110} - 2a_{001}a_{011}a_{110}a_{100} - 2a_{010}a_{011}a_{101}a_{100} \\
+ 4a_{000}a_{011}a_{101}a_{110} + 4a_{001}a_{010}a_{100}a_{111}.
\]

It does not take long to realize that \( \text{Det}(A) \) is a homogeneous polynomial for any choice of representations \( U, V, \) and \( W \). In fact its degree is \( 2(u_0 + u_1 - v_0 - v_1 + w_0 + w_1) \) in the notation of the
previous example. The polynomial is known to be irreducible, hence prime ($\mathbb{C}[a_{ijk}]$ is a UFD), so it generates a principal prime ideal, and hence it corresponds to an irreducible affine variety and also a projective variety.

Here is a fact: $\text{Det}(A) = 0$ if and only if there exists a nontrivial point $(x_0, x_1, y_0, y_1, z_0, z_1)$ such that all of the first partial derivatives of the function

$$\sum_{0\leq i,j,k\leq 1} a_{ijk}x_iy_jz_k$$

vanish simultaneously.

More generally hyperdeterminants are algebraic invariants under actions of tensor products of the special linear group that exist for matrices of certain formats. They are also slice invariants.

5 To be considered later

Here are some things I would like to think about:

1. What are possible multiranks of matrices? Is every finite sequence of integers a multirank of some matrix?

2. Following up on the previous point: can we interpret a multirank $(r_0, \ldots, r_{p-1})$ as a rank condition on a quiver representation? What about slice rank?

3. How can we formulate multidimensional matrix multiplication? (I have a pretty decent idea on how this can be done.) Is there an interesting multidimensional matrix group to be considered? (e.g. matrices of format $n \times n \times \cdots \times n$, and subgroups analogous to the classical matrix groups)

4. Is there a good canonical form for the slice matrices that outlines any relationships between them?

5. Now we have a few invariants for multidimensional matrices, and since we used rank to define them they are given by polynomial equations (hopefully homogeneous, remains to be seen). What can we say about varieties determined by rank conditions? Slice rank conditions? (Perhaps Grothendieck classes will say something about their geometry now.)

6. Following up on the previous point: how do we incorporate $T$-equivariance into the last example?

7. Following up on a previous point: given a matrix of a particular format, what slice conditions end up giving us a prime ideal/irreducible variety?

8. Is the condition that Cayley’s second hyperdeterminant vanishes $T$-equivariant?