Result of how Heuristic Risk Function leads to Quartile Search

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1 Context

The context of this problem is 20 Questions, or howver many questions. An oracle is thinking of a real number θ in the interval $[0, 1]$. A agent chooses a guess θ_i and queries the oracle. The oracle will respond saying that θ_i is too high or too low compared to θ. We denote those responses (labels) as -1 and $+1$ respectively.
Our model considers the case where there is a cost associated with each query, and that cost is *dependant on the label of the guess queried*. We can say, in the general case, that

$$\text{Cost}(-1) = \alpha, \quad \text{Cost}(+1) = \beta$$

The cost accumulates as the agent queries more θ_i's. The goal of this game is for the agent to query the oracle with guesses in such a way so that it *minimizes* the cost.

2 Relevant Model

To analyze this game, we are using the Bayesian probabilistic update model. This model considers *theta* as a random variable, with an initially uniform probability distribution in the domain $[0, 1]$. The agent will query the next θ_i based on this distribution, and the label for θ_i will change the probability distribution of θ according to Bayes' Rule.
If the first θ_i is chosen based on the uniform distribution of θ from $[0, 1]$, and if the verdict is $+1$, then the distribution will change so that *theta* will be uniformly distributed from $[\theta_i, 1]$. If the verdict is -1, the distribution will be from $[0, \theta_i]$. The range of possible values for θ shrinks at this iteration, and will continue to do so at every iteration of the game.
This approach has a lot of favorable properties. One benefit of this approach is that it also works for cases where the labels are noisy, and there is a probability p that the oracle was wrong in its verdicts. In that case, the probability distribution will not be uniform for a smaller and smaller range of possible θ's,

but will be piecewise constant, with some intervals being more probable than others, depending on the value of p.

3 Determining best θ_i

The one thing we did not address in the previous section is, given a general probability distribution for θ, how can we choose the $theta_i$'s appropriately to minimize the α, β cost.
Previous work on the subject has been focused on the case where the cost of a query is not dependant on the label, but is positive. In other words, the case that has been focused on is when $\alpha = \beta > 0$. In that case, it is shown by Horstein that taking the probabilistic median of the probability function serves as the optimal guess.
The reason for the optimality is that the median reduces the expected error between θ and θ_i.

$$\begin{aligned} R(\theta, \theta_i) &= \|\theta - \theta_i\| \\ E_\theta(R) &= g(\theta_i) \\ \theta_{best} &= argmin(g(\theta_i)) \end{aligned}$$

We will focus on the general case, where α and β may or may not be equal.

4 Heuristic Risk Function to Quartile Search

Firstly, we modified the risk function ($R(\theta, \theta_i)$) shown in previous section to include α and β terms. The new risk function is shown below.

$$R(\theta, \theta_i) = \begin{cases} \alpha(\theta_i - \theta) & \theta < \theta_i \\ \beta(\theta - \theta_i) & \theta > \theta_i \\ 0 & \theta = \theta_i \end{cases}$$

Proposition. If θ_i^* is equal to $argmin(E_\theta(R))$, then

$$\int_0^{\theta_i^*} f(\theta)\, d\theta = \frac{\beta}{\alpha + \beta}$$

where f is the probability density function of θ.

Proof. We take the expected value of the risk function with respect to θ.

$$\begin{aligned} E_\theta(R) = \int_0^1 R(\theta, \theta_i) f(\theta)\, d\theta &= \int_0^{\theta_i} \alpha(\theta_i - \theta) f(\theta)\, d\theta + \int_{\theta_i}^1 \beta(\theta - \theta_i) f(\theta)\, d\theta \\ &= -\int_0^{\theta_i} \alpha(\theta - \theta_i) f(\theta)\, d\theta + \int_{\theta_i}^1 \beta(\theta - \theta_i) f(\theta)\, d\theta \end{aligned}$$

We can express this in terms of anti-derivatives of f. To do that, we will use integration by parts. Let F be the anti-derivative of f. F will therefore be a CDF, with $F(0) = 0$ and $F(1) = 1$. Let $\mathbf{F}$ be the anti-derivative of F. We will start with the α term first.

$$\begin{aligned} -\alpha \int_0^{\theta_i} (\theta - \theta_i) f(\theta)\, d\theta &= -\alpha \left[(\theta - \theta_i) F(\theta)|_0^{\theta_i} - \int_0^{\theta_i} F(\theta)\, d\theta \right] \\ &= -\alpha \left[(0) F(\theta_i) + \theta_i F(0) - \mathbf{F}(\theta_i) + \mathbf{F}(0) \right] \\ &= \alpha \mathbf{F}(\theta_i) - \alpha \mathbf{F}(0) \end{aligned}$$

Now we can re-express the β term.

$$\begin{aligned} \beta \int_{\theta_i}^{1} (\theta - \theta_i) f(\theta)\, d\theta &= \beta \left[(\theta - \theta_i) F(\theta)|_{\theta_i}^{1} - \int_{\theta_i}^{1} F(\theta)\, d\theta \right] \\ &= \beta \left[(1 - \theta_i) F(1) - (0) F(\theta_i) + \mathbf{F}(\theta_i) - \mathbf{F}(1) \right] \\ &= \beta - \beta\theta_i + \beta \mathbf{F}(\theta_i) - \beta \mathbf{F}(1) \end{aligned}$$

We can now add the two expressions to make the expected risk.

$$E_\theta(R(\theta, \theta_i)) = g(\theta_i) = \alpha \mathbf{F}(\theta_i) - \alpha \mathbf{F}(0) + \beta - \beta\theta_i + \beta \mathbf{F}(\theta_i) - \beta \mathbf{F}(1)$$

To determine the *argmin*, we can take the derivative with respect to and set that equal to 0.

$$\frac{d}{d\theta_i} E_\theta(R(\theta, \theta_i)) = g'(\theta_i) = \alpha F(\theta_i) + \beta F(\theta_i) - \beta$$

To make sure that finding the critical points corresponds with finding a minimum, we take the second derivative and see if it is non-negative.

$$\frac{d^2}{d\theta_i^2} E_\theta(R(\theta, \theta_i)) = g''(\theta_i) = \alpha f(\theta_i) + \beta f(\theta_i) > 0$$

We assume α and β are non-negative, and f is non-negative because it is a pdf, so the second derivative is non-negative. We can thus say that the derivative evaluated at θ_i^*, which is $argmin(E_\theta(R))$, is equal to 0.

$$\alpha F(\theta_i^*) + \beta F(\theta_i^*) - \beta = 0$$

With some rearranging, we have

$$F(\theta_i^*) = \int_0^{\theta_i^*} f(\theta)\, d\theta = \frac{\beta}{\alpha + \beta} \quad \square$$