Geometry and Combinatorics of Matroids

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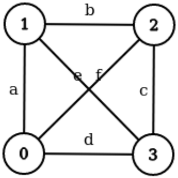
DIMACS REU, 2018

Example

Choose an edge such that when you pick any other edge you won't get a cycle. Then choose two edges... then three.. etc.



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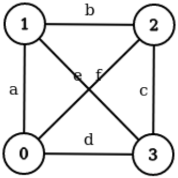

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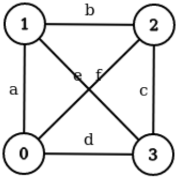

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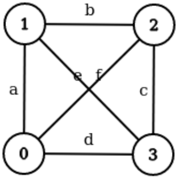

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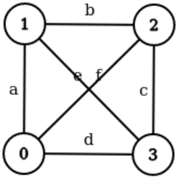

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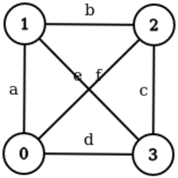

Special edge sets: (a),(b),(c),(d),(e),(f),(b,d),(a,c),(e,f)
(a,b,f),(c,f,d),(b,c,e),(a,d,e)
These "special" sets have a name: **flats**.

Example

Now we let these flats be represented by the variables:
$x_{(a)}, x_{(b)}, \ldots, x_{(a,d,e)}$

These group of variables form a polynomial ring
$S = \mathbb{R}[x_{(a)}, x_{(b)}, \ldots, x_{(a,d,e)}]$

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Now we let an ideal

$$\mathcal{I} = \left(\sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F, x_{F_1} x_{F_2} \right)$$

The ideal

So what does this even mean?

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and add the flats that **contain** the two flats you chose.

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Ex.
$i_1 = a$, then the first sum is:
$x_{(a)} + x_{(a,c)} + x_{(a,b,f)} + x_{(a,d,e)}$

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The difference of the two:
$x_{(a)} + x_{(a,c)} + x_{(a,d,e)} - x_{(b)} - x_{(b,d)} - x_{(b,c,e)}$

The ideal

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Ideal ends up with 60 elements.

Example

Now we construct a quotient ring from the polynomial ring and the ideal, and let it be defined by,

$$A = S/\mathcal{I}\left(\sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F, x_{F_1} x_{F_2}\right).$$

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A has a special name: **The Chow Ring**.
It's a tool from Algebraic Geometry.

Example

Now we want to get the generators from the Chow ring of the K_4 graph.

Which are:

1

$x_{(a)}, x_{(b,d)}, x_{(e,f)}, x_{(a,c)}, x_{(a,b,f)}, x_{(c,f,d)}, x_{(b,c,e)}, x_{(a,d,e)}$

$x^2_{(a,d,e)}$

Then count how many generators we have from each degree.

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$1, 8, 1$

Alright who cares...

So... why do this?

All graphs have a Chow ring, and every Chow ring of a graph ends up having a string of dimensions of various degree pieces that is palindromic.

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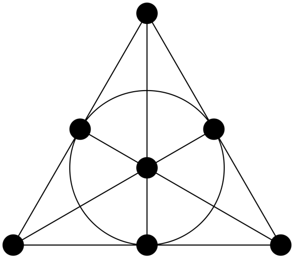
In reality, it's not just graphs, but every single matroid.

Quick Examples...



1,8,1

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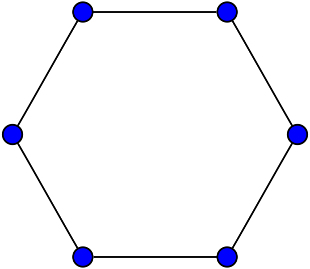

1,51,161,51,1

What's to come?

- Apply my program to different graphs
- Figure out how the pattern is made
- After that, move on to more abstract matroids and do the same for them

Thank you!

This project wouldn't have been possible without the support from the MAA/NREUP, through NSF grant DMS-1652506.

Citations

June Huh, *Combinatorial applications of the Hodge-Riemann relations*, Proceedings of the International Congress of Mathematicians 2018, to appear.

Oxley, J. G. *Matroid Theory*. Oxford University Press, 2011.