Geometry and Combinatorics of Matroids

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DIMACS REU, 2018
Example

Choose an edge such that when you pick any other edge you won’t get a cycle. Then choose two edges... then three.. etc.
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Special edge sets:

\[(a),(b),(c),(d),(e),(f),(b,d),(a,c),(e,f),(a,b,f),(c,f,d),(b,c,e),(a,d,e)\]

These “special” sets have a name: flats.
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These “special” sets have a name: flats.
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Now we let these flats be represented by the variables:

\[ x(a), x(b), \ldots, x(a,d,e) \]

These group of variables form a polynomial ring

\[ S = \mathbb{R}[x(a), x(b), \ldots, x(a,d,e)] \]
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Now we let an ideal
\[
I = \left( \sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F \cdot x_{F_1} \cdot x_{F_2} \right)
\]
The ideal

So what does this even mean?

$$\sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F$$
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\[ \sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F \]

Choose any two of the following flats: a,b,c,d,e,f and add the flats that \textbf{contain} the two flats you chose.
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Choose any two of the following flats: a,b,c,d,e,f and add the flats that contain the two flats you chose.

Ex.

\( i_1 = a \), then the first sum is:

\[ x(a) + x(a,c) + x(a,b,f) + x(a,d,e) \]
The ideal

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\[ i_1 = a, \text{ then the first sum is: } \]

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\[ i_2 = b, \text{ then the second sum is: } \]
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So what does this even mean?

\[ \sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F \]

Choose any two of the following flats: a,b,c,d,e,f and add the flats that **contain** the two flats you chose.

Ex.

\( i_1 = a \), then the first sum is:

\[ x(a) + x(a,c) + x(a,b,f) + x(a,d,e) \]

\( i_2 = b \), then the second sum is:

\[ x(b) + x(b,d) + x(a,b,f) + x(b,c,e) \]
The ideal

So what does this even mean?

\[ \sum_{i_1 \in F} X_F - \sum_{i_2 \in F} X_F \]

Choose any two of the following flats: a,b,c,d,e,f and add the flats that **contain** the two flats you chose.

Ex.

\( i_1 = a \), then the first sum is:

\[ X(a) + X(a,c) + X(a,b,f) + X(a,d,e) \]

\( i_2 = b \), then the second sum is:

\[ X(b) + X(b,d) + X(a,b,f) + X(b,c,e) \]

The difference of the two:

\[ X(a) + X(a,c) + X(a,d,e) - X(b) - X(b,d) - X(b,c,e) \]
Okay, what about $x_{F_1} x_{F_2}$?
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Choose 2 flats from the set of flats and multiply them, but not if one of the flats is contained in the other.
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$(a) \subseteq (a, c)$ so we don’t include $x_{(a)}x_{(a,c)}$ in the ideal.

Ideal ends up with 60 elements.
Now we construct a quotient ring from the polynomial ring and the ideal, and let it be defined by,

\[ A = S/\mathcal{I} \left( \sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F, x_{F_1} x_{F_2} \right). \]
Example

Now we construct a quotient ring from the polynomial ring and the ideal, and let it be defined by,

$$A = \frac{S}{\mathcal{I}} \left( \sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F, x_F x_{F_1} x_{F_2} \right).$$

A has a special name: The Chow Ring. It’s a tool from Algebraic Geometry.
Now we want to get the generators from the Chow ring of the $K_4$ graph.

Which are:

1

$x(a)$, $x(b,d)$, $x(e,f)$, $x(a,c)$, $x(a,b,f)$, $x(c,f,d)$, $x(b,c,e)$, $x(a,d,e)$

$x^2(a,d,e)$

Then count how many generators we have from each degree.
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Which are:

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$x(a), x(b,d), x(e,f), x(a,c), x(a,b,f), x(c,f,d), x(b,c,e), x(a,d,e)$

$x^2(a,d,e)$

Then count how many generators we have from each degree.

1, 8, 1
Alright who cares...

So... why do this?

All graphs have a Chow ring, and every Chow ring of a graph ends up having a string of dimensions of various degree pieces that is palindromic.
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All graphs have a Chow ring, and every Chow ring of a graph ends up having a string of dimensions of various degree pieces that is palindromic.

In reality, it’s not just graphs, but every single matroid.
Quick Examples...

1, 8, 1

Diagram with interconnected points forming a triangular structure.
Quick Examples...

1, 51, 161, 51, 1
What’s to come?

- Apply my program to different graphs
- Figure out how the pattern is made
- After that, move on to more abstract matroids and do the same for them
Thank you!

This project wouldn’t have been possible without the support from the MAA/NREUP, through NSF grant DMS-1652506.