Geometry and Combinatorics of Matroids

Froylan Maldonado
Mentor: Dr. Nicola Tarasca

San Diego City College

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What is a Mixed Discriminant?

To define the mixed discriminant , let $\vec{A} = (A_1, ..., A_d)$ be a collection of d x d matrices, and consider the function

$$det_{\vec{A}} : \mathbb{R}^d \to \mathbb{R}, \qquad (t_1, ..., t_d) \longmapsto det(t_1 A_1 + ... + t_d A_d)$$

which is a homogeneous polynomial of degree d. The number

$$D(A_1, ..., A_d) = \frac{\partial^d}{\partial t_1 ... \partial t_d} det_{\vec{A}}(t_1, ...t_d)$$

is called the *mixed discriminant* of $\vec{A}$. The mixed discriminant is symmetric in $\vec{A}$, and it is nonnegative whenever all the matrices in $\vec{A}$ are positive semidefinite.

Example

Let $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$det(xA_1 + yA_2 + zA_3) = \begin{vmatrix} x+y+z & 0 & 0 \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$

$$D(A_1, A_2, A_3) = \frac{\partial^3}{\partial x \partial y \partial z}(x+y+z)^3 = \mathbf{6}$$

HR(P)

Let $\vec{P} = (P_1, ..., P_{d-2})$ be a collection of d x d positive semidefinite matrices.

$$HR(\vec{P}) : Sym_d \times Sym_d \to \mathbb{R}, \qquad (\eta_1, \eta_2) \mapsto D(\eta_1, \eta_2, P_1, ..., P_{d-2})$$

HR(P)

HR($\vec{P}$) can be represented by a matrix T.

$$T = [D(\eta_i, \eta_j, P_1, ...P_{d-2})]$$

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Theorem 1

Any matrix representing HR($\vec{P}$) has exactly one positive eigenvalue.

Goals

Long term: To show how the general theorem of matroids can be applied in various different cases, and to have a few examples to show Rota's unimodality conjecture: If $w_k(M)$ is the number of rank k flats of a rank d matroid M, then the sequence $w_0(M), ..., w_d(M)$ is unimodal. Welsh conjectured more generally that the sequence is log-concave.

Right now: Prove that the determinant of a matrix representing one case of HR($\vec{P}$) is nonpositive.

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Citations

June Huh, *Combinatorial applications of the Hodge-Riemann relations*, Proceedings of the International Congress of Mathematicians 2018, to appear.