Research Question Presentation on the Edge Clique
Covers of a Complete Multipartite Graph

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Suppose we have a graph $G$.

A **clique** $S$ of $G$ is a subset of $V(G)$ such that any two vertices of $S$ are adjacent in $G$.

A **vertex clique cover** of $G$ is a family of cliques such that any arbitrary vertex of $G$ is contained in some clique in the family.

**Example:**

Some vertex clique covers of $G$:

1) $\{ (V_1, V_4), (V_2, V_3, V_5) \}$
2) $\{ (V_1, V_2), (V_3, V_4, V_5) \}$
3) $\{ (V_1, V_2), (V_4, V_5), (V_3) \}$
4) $\{ (V_2, V_3, V_5), (V_3, V_4, V_5), (V_1) \}$
An *edge clique cover* of G is a family of cliques such that any arbitrary edge of G is covered by some clique in the family.
(if an edge is covered by a clique, both of its incident vertices are present in that clique, as cliques only contain vertices and not edges)

Example:

Some edge clique covers of G:
1. \{ (V_2, V_3, V_5), (V_3, V_4, V_5), (V_1, V_2), (V_1, V_4) \}
2. \{ (V_2, V_3, V_5), (V_3, V_4), (V_4, V_5), (V_1, V_2), (V_1, V_4) \}
3. \{ (V_3, V_4, V_5), (V_2, V_3), (V_2, V_5), (V_1, V_2), (V_1, V_4) \}

The *vertex clique cover number* of a graph G, denoted \( \theta_v(G) \), is the minimum size of vertex clique covers of G.
The *edge clique cover number* of a graph G, denoted \( \theta_e(G) \), is the minimum size of edge clique covers of G.
Complete Multipartite Graphs:

A *complete graph*, denoted $K_n$, is a graph $G$ with $n$ vertices where every vertex of $G$ is connected to every other vertex. 

Examples:

$K_3$:

\[\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{complete_graph_3.png}} \\
\end{array}\]

$K_4$:

\[\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{complete_graph_4.png}} \\
\end{array}\]

A *multipartite graph* is a graph $G$ that is composed of partitions of $V(G)$ where there are no edges within each partition.

Example:

\[\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{multipartite_graph.png}} \\
\end{array}\]

A *complete multipartite graph* is a multipartite graph where every vertex in each partite site is connected to every other vertex in all the other partite sets.

Example:

\[\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{complete_multipartite_graph.png}} \\
\end{array}\]
If the amount of vertices in each partition differ, then we denote the complete multipartite graph on $r$ partite sets as $K_{n_1,n_2,\ldots,n_r}$, where $n_1 \geq n_2 \geq \ldots \geq n_r$.

If $n_1 = n_2 = \ldots = n_r$, then we denote the complete multipartite graph on $r$ partite sets with $n$ vertices in each set as $K^r_n$. A vertex in $K^r_n$ is denoted as $v^i_j$, where $1 \leq i \leq r$ and $1 \leq j \leq n$. 
Note that $\theta_v(K_{n_1,n_2,\ldots,n_r})=n_1$ is a known result, where $n_1 \geq n_2 \geq \ldots \geq n_r$.

Proof: Each vertex in the largest set, $n_1$, cannot be connected with another vertex in $n_1$ by definition, so they must all be contained in distinct cliques. Thus, $\theta_v(K_{n_1,n_2,\ldots,n_r}) \geq n_1$.

Compose cliques as such:

for each $1 \leq i \leq n_1$, let $S_i = \{v_i^j \mid 1 \leq j \leq r\}$.
Then $\{S_1,S_2,\ldots,S_n\}$ is an edge clique cover of $K_{n_1,n_2,\ldots,n_r}$.
(For example, $S_1=\{v_1^1, v_1^2, \ldots, v_1^r\}$, $S_2=\{v_2^1, v_2^2, \ldots, v_2^r\}$, \ldots, $S_n=\{v_n^1, v_n^2, \ldots, v_n^r\}$)

This process illustrates that $\theta_v(K_{n_1,n_2,\ldots,n_r}) \leq n_1$.

$\therefore \theta_v(K_{n_1,n_2,\ldots,n_r}) = n_1$ ■

For this project, we would like to focus on $\theta_e(K_{n_1,n_2,\ldots,n_r})$.

Determining the edge clique cover number of a complete multipartite graph is a much more difficult task.
It is well known that $\theta_e(k_{n_1,n_2,\ldots,n_r}) \geq n_1 n_2$. However, we wish to know a more precise value of $\theta_e(k_{n_1,n_2,\ldots,n_r})$. We look at several cases for the value of $r$, where $n_1 \geq n_2 \geq \ldots \geq n_r$. 
Several known results:

1) \( \theta_e(k_n^2) = n^2 \)

\[
\begin{align*}
\text{there are a total of } n^2 \text{ edges in this graph and each edge belongs to a unique clique since there are no other connections other than the edges from one partite set to the other.}
\end{align*}
\]

2) \( \theta_e(k_n^3) = n^2 \)
3) When $n$ is a power of a prime, $\theta_e(k_n^4) = n^2$

Example:
Suppose $n = 4$ and $r = 4$. We choose our cliques as such:

\{\nu_1^1, \nu_2^2, \nu_3^3, \nu_4^4\} \{\nu_2^1, \nu_2^2, \nu_3^3, \nu_2^4\} \{\nu_3^1, \nu_1^2, \nu_3^3, \nu_3^4\} \{\nu_4^1, \nu_2^2, \nu_3^3, \nu_4^4\}

\{\nu_1^1, \nu_2^2, \nu_3^3, \nu_4^4\} \{\nu_2^1, \nu_2^2, \nu_4^3, \nu_1^4\} \{\nu_3^1, \nu_2^2, \nu_3^3, \nu_4^4\} \{\nu_4^1, \nu_2^2, \nu_1^3, \nu_3^4\}

\{\nu_1^1, \nu_2^2, \nu_3^3, \nu_4^4\} \{\nu_2^1, \nu_3^2, \nu_1^3, \nu_4^4\} \{\nu_3^1, \nu_3^2, \nu_2^3, \nu_1^4\} \{\nu_4^1, \nu_3^2, \nu_4^3, \nu_2^4\}

\{\nu_1^1, \nu_4^2, \nu_3^3, \nu_4^4\} \{\nu_2^1, \nu_4^2, \nu_2^3, \nu_3^4\} \{\nu_3^1, \nu_4^2, \nu_1^3, \nu_2^4\} \{\nu_4^1, \nu_4^2, \nu_3^3, \nu_1^4\}

Any edge of $k_n^4$ is covered by one of the above 16 cliques. Thus, it is a minimum edge clique cover of $K_4^4$. 
A **Latin Square** of order \( n \) is an \( n \times n \) rectangular array where each row and column includes all the integers from 1 to \( n \). We denote the entry in the \( i^{th} \) row and \( j^{th} \) column, where \( i, j \in \{1,2,\ldots,n\} \), as \( L(i,j) \).

Two Latin Squares \( L_1 \) and \( L_2 \) are said to be **orthogonal** if for any \( i, j \in \{1,2,\ldots,n\} \), \( \exists k, l \in \{1,2,\ldots,n\} \) such that \( L_1(k,l) = i \) and \( L_2(k,l) = j \).

Example: Suppose \( n = 5 \). Then the following two Latin squares are orthogonal.

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & 1 \\
3 & 4 & 5 & 1 & 2 \\
4 & 5 & 1 & 2 & 3 \\
5 & 1 & 2 & 3 & 4 \\
\end{array}
\quad
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
3 & 4 & 5 & 1 & 2 \\
5 & 1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 & 1 \\
4 & 5 & 1 & 2 & 3 \\
\end{array}
\]
$L(n)$ is defined to be the maximum size of mutually orthogonal Latin Squares.

It is known that for any integer $n$, $L(n) \leq n - 1$ and if $n$ is a prime power, then $L(n) = n - 1$.

If $m \leq L(n) + 2$, where $m$ is the amount of sets of a complete multipartite graph, then we can construct an edge clique cover for $k_m^n$ by looking at our set of mutually orthogonal Latin Squares. Each clique has $m$ entries where $i, j \in \{1, 2, \ldots, n\}$ are the first and second entries in the clique, $L_1(i, j)$ is the third, ...., and $L_{n-1}(i, j)$ is the $m^{th}$ entry. We will end up with a total of $n^2$ such cliques since there are a total of $n^2$ ordered pairs $(i, j)$. 
For example, we can come up with $\theta_e(k^3_n)$ by examining a Latin Square of order $n$.

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</table>

For the first $n$ cliques, we observe all $1,j$ values in the first row. We find cliques similarly for values in all $n$ rows.

- $\{1,1,1\}$
- $\{1,2,2\}$
- $\{1,3,3\}$
- $\{2,1,2\}$
- $\{2,2,3\}$
- $\{2,n-1,n\}$
- $\{2,n,1\}$
- $\{3,1,3\}$
- $\{3,n-2,n\}$
- $\{3,n-1,1\}$
- $\{3,n,2\}$
- $\{n,1,n\}$
- $\{n,2,1\}$
- $\{n,3,2\}$
- $\{n,4,3\}$

: $\{n,1,n\}$

Thus, we have:

- $\theta_e(k^3_n)$

By examining the Latin Square, we can find the cliques as shown in the table.
Now suppose we have two orthogonal Latin Squares, $L_1$ and $L_2$, of order 5.

Since we have two orthogonal Latin Squares, $m \leq 2 + 2 = 4$.

So we can find $\theta_e(k^4_n)$ by forming the following cliques:

Row $i$ Column $j$ $L_1(i,j)$ $L_2(i,j)$

{1,1,1,1}, {1,2,2,2}, {1,3,3,3}, {1,4,4,4}, {1,5,5,5}, {2,1,2,3},
{2,2,3,4}, {2,3,4,5}, {2,4,5,1}, {2,5,1,2}, etc.

Since there are $n^2$ entries in the Latin Squares, and we form another clique for each entry, we have $\theta_e(k^4_n) = n^2$
In this project, we will focus on $\theta_e(K_{n_1,n_2,\ldots,n_r})$.

Research questions on $\theta_e(K_{n_1,n_2,\ldots,n_r})$:
1) Suppose $n_1 = n_2 = \ldots = n_r$. Can we find better bounds for the edge clique cover number of any complete multi-partite graph $K^m_n$ by looking at sets of mutually orthogonal Latin Squares?
2) More specifically, what are the values of $\theta_e(k^3_n)$ or $\theta_e(k^4_n)$?
3) What conditions can we put on $n_1, n_2, \ldots, n_r$ so that $\theta_e(K_{n_1,n_2,\ldots,n_r}) = n_1 n_2$?