Unique Decomposability of Graph Gluings

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Can the components of a multi-component graph be ‘glued’ together in such a way that the resultant graph is ‘uniquely decomposable’ into a collection of graphs similar to the original components?
Definition (B. DeMarco and A. Redlich)

A tuple \((G_1, G_2, \ldots, G_k, H, H_1, H_2, \ldots, H_k)\) is a gluing of \(G_1, \ldots, G_k\) if:

- \(H\) is a connected graph,
- \(H_1, \ldots, H_k\) are subgraphs of \(H\),
- \(H_i \sim G_i\) for all \(i \in \{1, 2, \ldots, k\}\), and
- \(\bigcup_{i=1}^k E(H_i) = E(H)\).
Gluing of Graphs

**Definition (B. DeMarco and A. Redlich)**

A tuple \((G_1, G_2, \ldots, G_k, H, H_1, H_2, \ldots, H_k)\) is a *gluing* of \(G_1, \ldots, G_k\) if:

- \(H\) is a connected graph,
- \(H_1, \ldots, H_k\) are subgraphs of \(H\),
- \(H_i \sim G_i\) for all \(i \in \{1, 2, \ldots, k\}\), and
- \(\bigcup_{i=1}^{k} E(H_i) = E(H)\).

**Example:**

\[\begin{align*}
G_1 &\quad \rightarrow \quad H \\
G_2 &\quad \rightarrow \quad H
\end{align*}\]
Definition (B. DeMarco and A. Redlich)

A tuple \((G_1, \ldots, G_k, H)\) is *uniquely decomposable* if there exists only one tuple \(H_1, \ldots, H_k\) such that \((G_1, \ldots, G_k, H, H_1, \ldots, H_k)\) is a gluing.
Unique Decomposition

Definition (B. DeMarco and A. Redlich)

A tuple \((G_1, \ldots, G_k, H)\) is \textit{uniquely decomposable} if there exists only one tuple \(H_1, \ldots, H_k\) such that \((G_1, \ldots, G_k, H, H_1, \ldots, H_k)\) is a gluing.

Example 1:

\[ \begin{align*}
G_1 & = \{1, 2\} \\
G_2 & = \{2, 3, 4, 5, 6\}
\end{align*} \]
**Definition (B. DeMarco and A. Redlich)**

A tuple \((G_1, \ldots, G_k, H)\) is *uniquely decomposable* if there exists only one tuple \(H_1, \ldots, H_k\) such that \((G_1, \ldots, G_k, H, H_1, \ldots, H_k)\) is a gluing.

**Example 1:**

![Diagram](image)
Definition (B. DeMarco and A. Redlich)

A tuple \((G_1, \ldots, G_k, H)\) is *uniquely decomposable* if there exists only one tuple \(H_1, \ldots, H_k\) such that \((G_1, \ldots, G_k, H, H_1, \ldots, H_k)\) is a gluing.

Example 1:

\[
\begin{array}{c}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\]

\[
\begin{array}{c}
G_1 & G_2 \\
\end{array}
\]

\[
\begin{array}{c}
H \\
\end{array}
\]

\((G_1, G_2, H)\) not uniquely decomposable:
Definition (B. DeMarco and A. Redlich)

A tuple \((G_1, \ldots, G_k, H)\) is *uniquely decomposable* if there exists only one tuple \(H_1, \ldots, H_k\) such that \((G_1, \ldots, G_k, H, H_1, \ldots, H_k)\) is a gluing.

Example 1:

\[(G_1, G_2, H)\] not uniquely decomposable:

- \(H_1 = \{1, 2\}\) and \(H_2 = \{2, 3, 4, 5, 6\}\)
- or
- \(H_1 = \{5, 6\}\) and \(H_2 = \{1, 2, 3, 4, 5\}\)
Unique Decomposition

Definition (B. DeMarco and A. Redlich)

A tuple \((G_1, \ldots, G_k, H)\) is *uniquely decomposable* if there exists only one tuple \(H_1, \ldots, H_k\) such that \((G_1, \ldots, G_k, H, H_1, \ldots, H_k)\) is a gluing.

Example 2:

\[ \begin{array}{c}
G_1 \\
\end{array} \quad \begin{array}{c}
G_2 \\
\end{array} \]

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Graph Gluings
Definition (B. DeMarco and A. Redlich)

A tuple \((G_1, \ldots, G_k, H)\) is *uniquely decomposable* if there exists only one tuple \(H_1, \ldots, H_k\) such that \((G_1, \ldots, G_k, H, H_1, \ldots, H_k)\) is a gluing.

Example 2:

\[ G_1 \quad H \quad G_2 \]

\[
\begin{align*}
G_1 & : 1 \rightarrow 2 \rightarrow 3 \\
G_2 & : 4 \rightarrow 5 \\
H & : 6
\end{align*}
\]
Unique Decomposition

Definition (B. DeMarco and A. Redlich)

A tuple \((G_1, \ldots, G_k, H)\) is **uniquely decomposable** if there exists only one tuple \(H_1, \ldots, H_k\) such that \((G_1, \ldots, G_k, H, H_1, \ldots, H_k)\) is a gluing.

Example 2:

\(G_1\)

\(G_2\)

\((G_1, G_2, H)\) not uniquely decomposable:
Definition (B. DeMarco and A. Redlich)

A tuple \((G_1, \ldots, G_k, H)\) is *uniquely decomposable* if there exists only one tuple \(H_1, \ldots, H_k\) such that \((G_1, \ldots, G_k, H, H_1, \ldots, H_k)\) is a gluing.

Example 2:

\[(G_1, G_2, H)\] not uniquely decomposable:

\[H_1 = \{4, 5\}\] and \[H_2 = \{1, 2, 3, 4, 6\}\]

or

\[H_1 = \{4, 6\}\] and \[H_2 = \{1, 2, 3, 4, 5\}\]
**Unique Decomposition**

**Definition (B. DeMarco and A. Redlich)**

A tuple \((G_1, \ldots, G_k, H)\) is *uniquely decomposable* if there exists only one tuple \(H_1, \ldots, H_k\) such that \((G_1, \ldots, G_k, H, H_1, \ldots, H_k)\) is a gluing.

Example 3:

\[
\begin{align*}
G_1 &= \\
\text{and} \quad G_2 &= 
\end{align*}
\]
A tuple \((G_1, \ldots, G_k, H)\) is *uniquely decomposable* if there exists only one tuple \(H_1, \ldots, H_k\) such that \((G_1, \ldots, G_k, H, H_1, \ldots, H_k)\) is a gluing.

Example 3:
Definition (B. DeMarco and A. Redlich)

A tuple \((G_1, \ldots, G_k, H)\) is uniquely decomposable if there exists only one tuple \(H_1, \ldots, H_k\) such that \((G_1, \ldots, G_k, H, H_1, \ldots, H_k)\) is a gluing.

Example 3:

\[(G_1, G_2, H)\) is uniquely decomposable:
Definition (B. DeMarco and A. Redlich)

A tuple \((G_1, \ldots, G_k, H)\) is uniquely decomposable if there exists only one tuple \(H_1, \ldots, H_k\) such that \((G_1, \ldots, G_k, H, H_1, \ldots, H_k)\) is a gluing.

Example 3:

\((G_1, G_2, H)\) is uniquely decomposable:

\(H_1 = \{3, 6\}\) and \(H_2 = \{1, 2, 3, 4, 5\}\)
Theorem (DeMarco and Redlich)

If $G_1 \neq G_2$, neither $G_1$ nor $G_2$ is a single vertex, \[ \{G_1, G_2\} \neq \{P_1, P_2\} \quad \text{and} \quad \{G_1, G_2\} \neq \{P_1, P_3\}, \]
there exists a graph $H$ such that $(G_1, G_2, H, H_1, H_2)$ is a uniquely decomposable gluing and $H \neq G_1, G_2$. Furthermore, $H$ may be constructed explicitly.
Theorem (DeMarco and Redlich)

If $G_1 \neq G_2$, neither $G_1$ nor $G_2$ is a single vertex, \{G_1, G_2\} \neq \{P_1, P_2\}$ and \{G_1, G_2\} \neq \{P_1, P_3\}, there exists a graph $H$ such that $(G_1, G_2, H, H_1, H_2)$ is a uniquely decomposable gluing and $H \neq G_1, G_2$. Furthermore, $H$ may be constructed explicitly.

So, the only (interesting) two-component graphs that do not admit a unique composition are:

\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{and}
\end{array}
\end{array}
\end{align*}
Goals for the Summer

- Investigate the gluing of multi-component graphs
  - 3-component graphs
  - Graphs whose components are trees
- Understand the context of this problem
  - Random graph theory