On the Optimal Starting State for a Deterministic Scan in the Three-Color Potts Model

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The setting

- We start with a (not necessarily proper) coloring c_0 of an undirected graph G using three colors, and a given sequence $v_1 \ldots, v_T$ of vertices (which is called a "deterministic scan," as opposed to a sequence of randomly selected vertices).
- At time $t \in \{1, \ldots T\}$, we update the color of v_t - we look at its neighborhood, and if it has x blue neighbors, the probability of the new color of v_t being blue is proportional to λ^x, where $\lambda > 1$ is some constant, and similarly for the other two colors.

Problem (Lubetzky):

Which choice of the initial coloring c_0 makes the event that at time T, all vertices are colored blue, most likely?

- The obvious answer seems to be color everything blue - this is true for two colors.

Generalised setting (Narayanan)

- Consider a finite, dynamic graph $G_t = (V, E_t)$. Our random coloring process is a (non-time-homogeneous) Markov chain (X_t), with state space Q^V, where Q is a finite set of colors. Let F_t be an increasing function on $\mathbb{N}$ for each t.
- At time t, the chain moves as follows: pick a vertex v_t, and for a coloring $x \in Q^V$ take the set

$$S(x, v_t) = \left\{ y \in Q^V : \; y(w) = x(w) \; \forall \; w \neq v \right\}.$$

Then, the transition probability $P_t(x, y) = P[X_{t+1} = y | X_t = x]$ is

$$P_t(x, y) = \begin{cases} \frac{F_t\left(N^x_{y(v_t)}(v_t)\right)}{Z(x, v_t, F_t)} & \text{if } y \in S(x, v_t) \\ 0 & \text{else,} \end{cases}$$

where $N^x_y(v_t)$ is the number of neighbors of v_t with color $y(v_t)$ in x, and $Z(x, v_t, F) = \sum_{q \in Q} F\left(N^x_q(v_t)\right)$.

The questions

Claim 1:

For any choice of dynamic graph G_t, update sequence $\{v_t\}$ and sequence of updating functions $\{F_t\}$, where each F_t is increasing, the optimal starting coloring c_0 for maximising the probability of every vertex being blue/a given vertex being blue/etc. at the stopping time T is the all blue coloring.

- This is false - i.e. there exists a counterexample.

Claim 2:

Does adding the requirement of each F_t being convex help?

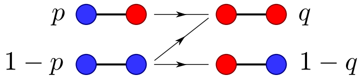
- This is also false.
- We can further strengthen the requirements by requiring G_t and/or F_t to not change over time - we also found counterexamples in those cases, which are closer to the initial setting where F_t is an exponential.

(Sketch of) The proof for two colors

- Our claim is that if we have two initial colorings that differ only in the color of one vertex w, then the coloring that has w blue is at least as likely to end up in all blue as the other.
- For a sequence of length one this is trivial.
- Now suppose the claim is true for a sequence of length n, and add another update of a vertex v to the beginning of the sequence to form a sequence of length $n + 1$.
- We have that the initial coloring with w blue leads to a (not necessarily strictly) higher probability of v being updated to blue in that first update, by the base case, and it also leads to higher conditional probabilities of getting all blue at the end given that v updates to either red or blue in the first update, by the induction hypothesis. It is easy to see that the initial coloring then leads to a higher probability of all blue at the end.
- By recoloring vertices one by one, we see that all blue is the best initial coloring.

Picture of proof for two and more details

- Here we can see two branches when we update vertex v (assume that it is a neighbor of v_1).
- On both sides there are written probabilities that each of the configurations happens.
- The arrows between the sides imply that from induction we know that the left one is more likely to end in all blue than the right one.
- The last thing which is missing is to say that $1 - p \geq 1 - q$. Then we can "substract" $1 - q$ probability from the *both blue* state. As a result of the induction assumption we know that *both blue* state is more likely to end up in all blue state than the two on the right side (as arrows show).
- The last part is that $1 - p - (1 - q) + p = q$ and both states on the left are more probable than the *both red* one.

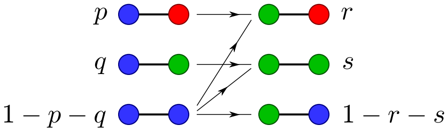


The proof where all blue works for three colors with enough blue vertices

Claim:

If we use convex updates and every time we update a vertex, we know that it has at least as many blue neighbors as any other color, then we can prove that starting with all blue is the best.

- For example one way how to assure there is always more blue neighbours is by taking an arbitrary graph with max degree k, adding k blue vertices, and connecting them to all other vertices.
- Let's try to use the same idea of proof which we used for 2-color problem.



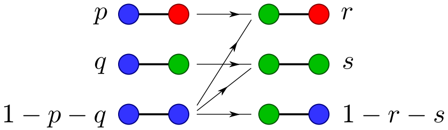
- We know that $s \geq q$ because our function is increasing. So if we could assure that also $r \geq p$, then we could use the same proof as previously for two colors.
- We can express the condition $r \geq p$ as functional equation:

$$\frac{f(R)}{f(R)+f(G)+f(B)} \leq \frac{f(R)}{f(R)+f(G+1)+f(B-1)}$$

- This can be rewritten as:

$$f(G+1)-f(G) \leq f(B)-f(B-1)$$

- And because of the convexity and majority of blue vertices we know that the inequality holds.



Counterexamples

- However, this is about all we can say in the positive direction for three colors. We have counterexamples in a wide range of cases.

Tools

Randomization: Color a vertex randomly (using $F = 1$, or $F(0) = 1, F(1) = 1 + \epsilon$)

Copying: Duplicate a vertex by using $F(0) = 1, F(1) = 10^6$

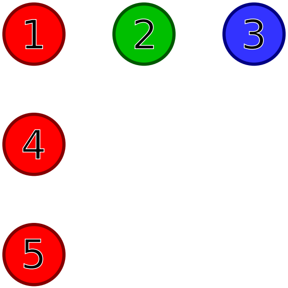
WTAOTWT (Winner Take All or Three Way Tie): Use $F(0) = 1, F(1) = 1, ..., F(n-1) = 1, F(n) = 10^6$.

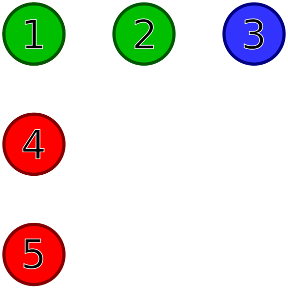
Dynamic Graph Counterexample with Convex Updates

- There are five vertices, V1, V2, V3, V4, V5, and we want to maximize the probability that V1 is blue at the end (which we could easily replace with all vertices blue).
- The two initial states we consider are all vertices blue, and all vertices blue except V1 red.
- Steps 1, 2: Update V2 and V4 randomly, using $F = 1$.
- Step 3: Make V5 agree with V4, using $F(0) = 1, F(1) = 10^6$.
- Step 4: Update V1 based on V1, V2, V3, using $F(0) = 1, F(1) = 1, F(2) = 10^6, \ldots$
- Step 5: Update V1 based on V1, V2, V3, V4, V5, using $F(0) = 1, F(1) = 1, F(2) = 1, F(3) = 10^6, \ldots.$
- Step 6: Update V1 based on V1, V2 using $F(0) = 1, F(1) = 1, F(2) = 10^6, \ldots.$

Visualization of Convex Counterexample

It turns out that the only important case is where V2 gets updated to green and V4 and V5 get updated to red:

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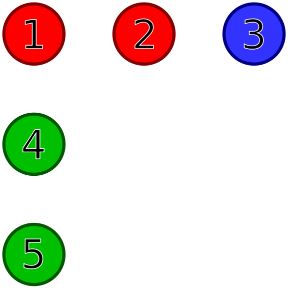
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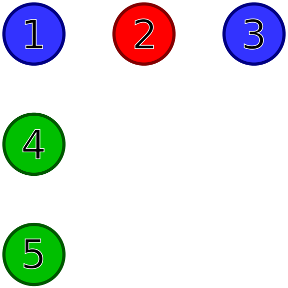
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Single Update Function?

- How can we adapt this to use the same update function at every step? We have to implement randomization, copying, and WTAOTWT with one function.
- Randomization: Use $F(1) - F(0) << F(0)$.
- Copying: Use $F(2) - F(1) >> F(1) - F(0)$ and amplify by using many intermediary vertices.
- WTAOTWT: Can implement using randomization and copying (due to Professor Narayanan): multiply neighbors by $\lambda = \frac{3}{4}$ and make all the new neighbors the same random color. Then if there was a single clear winner among the old neighbors, that will still win; otherwise, the random color will win.

Summary of Counterexamples

- Counterex to coupling for three colors
- Dynamic graph counterex, with (increasing) functions changing in time and the final event that a particular vertex is blue.
- Fixed graph, with the final event that all vertices are blue.
- Convex functions (discussed above).
- Fixed convex function on dynamic graph
- Fixed convex function on fixed graph
- Fixed exponential function on dynamic graph
- Changing exponential function on fixed graph

Conclusion

- We've constructed a sequence of better and better counterexamples, leading up to counterexamples when either the function is a fixed exponential and the graph changes over time, or the function is a changing exponential and the graph is fixed. Combining these would answer the original question!
- But there is a major issue when we use a fixed graph and a fixed exponential function: we need the degrees to be small enough that we can exploit $F(1) - F(0) \approx 0$ to create randomly colored vertices, but we need the degrees to be large enough that the values of F respond to changes in counts, and this is surprisingly difficult when $\frac{F(n)}{F(n-1)}$ is constant.

Thank you!

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