Competition Graphs and Permutation Patterns

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Overview

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Competition Graphs

- Given digraph $D$, its **competition graph** $C(D)$ has:
  - The same vertex set $V(D)$
  - Edge between $u$ and $v$ if $D$ contains arcs $(u, w)$ and $(v, w)$
The **doubly partial order** on $\mathbb{R}^2$:

- $S = \{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)\}$
- $(x_i, y_i) \prec (x_j, y_j)$ if $x_i < x_j$ and $y_i < y_j$

A **permutation** is a sequence $\pi = \pi_1\pi_2\ldots\pi_n$, where each $\pi_i$ is distinct and $\in \{1, 2, \ldots, n\}$

- $S_n$ is the set of all permutations of length $n$
- A pattern $\tau \in S_k$, $k < n$, is the perm. on a subsequence of $\pi \in S_n$
- Ex: $\pi = 5634127$ contains the pattern 123 but avoids 132
- $S_n(\tau)$ is the set of $\pi \in S_n$ that avoid $\tau$
The Competition Graph of a Permutation

- Given \( \pi \in S_n \), \( D(\pi) \) is the digraph given by the doubly partial order.
- \( C(D(\pi)) \) is its competition graph.
  - For simplicity, use \( C(\pi) \) instead of \( C(D(\pi)) \)
- From left to right: \( \pi = 31452 \), \( D(\pi) \), and \( C(\pi) \)

Edges in \( C(\pi) \) correspond to 123, 132 patterns!
Some More Notation

- Graph $G \in C(S_n)$, if there exists $\pi \in S_n$ such that $C(\pi)$, with the isolated vertices removed, is isomorphic to $G$
  - Analogously define $C(S_n(123))$ and $C(S_n(132))$
- If $\pi \in S_n$ is in the set $C_n^{-1}(G)$, $C(\pi)$, with the isolated vertices removed, is isomorphic to $G$
  - For instance, $31452 \in C_5^{-1}(K_3)$

Analogously define $C_n^{-1}(G, \tau)$, which restricts permutations to $S_n(\tau)$

Let $c_n(G) = |C_n^{-1}(G)|$ and $c_n(G, \tau) = |C_n^{-1}(G, \tau)|$
Forbidden Induced Subgraphs

- Take any $G \in C(S_n)$:
  - If $G$ contains $P_3$ as an induced subgraph, $G \notin C(S_{132})$
  - If $G$ contains $K_{1,3}$ as an induced subgraph, $G \notin C(S_{123})$

- In other words (or notation):
  - $C^{-1}(P_3, 132) = \emptyset$ (No permutations in $S_n(132)$ can form $P_3$)
  - $C^{-1}(K_{1,3}, 123) = \emptyset$ (No permutations in $S_n(123)$ can form $K_{1,3}$)

Conjecture: Given that $G \in C(S_n)$:
- $G \notin C(S_{132})$ iff $P_3$ is an induced subgraph
- $G \notin C(S_{123})$ iff $K_{1,3}$ is an induced subgraph
Bijections

- From the previous slide:
  - \( C^{-1}(P_3, 132) = \emptyset \)
  - \( C^{-1}(K_{1,3}, 123) = \emptyset \)

- Can we do better?
  - \( C^{-1}(P_m, 132) = \emptyset \)
  - \( C^{-1}(K_{1,3}, 123) = \emptyset \)

- We can do even better!
  
  \[
  C^{-1}(P_3, 123) \iff C^{-1}(K_{1,3}, 132) \\
  C^{-1}(P_m, 123) \iff C^{-1}(K_{1,m}, 132) \\
  C^{-1}(P_3) \iff C^{-1}(K_{1,3}) \\
  C^{-1}(P_m) \iff C^{-1}(K_{1,m})
  \]

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Counting

- Formula for $c_n(K_{1,3}, 132), c_n(K_{1,3})$ using induction

  $$c_n(K_{1,3}, 132) = \sum_{k=1}^{n-6} [(k - 1) \cdot 2^k + 1]$$

  $$c_n(K_{1,3}) = 4 \cdot \sum_{k=1}^{n-6} [(k - 1) \cdot 2^k + 1]$$

- Problem: Induction doesn’t generalize to $c_n(K_{1,m}, 132), c_n(K_{1,m})$

- A different way to compute $c_n(K_{1,m}, 132), c_n(K_{1,m})$
  - Insertion approach
  - Does not rely on induction
  - Problem: Insertion is conceptually easy but difficult in practice
Solution: a recurrence relation:

\[ c_n(K_{1,m}, 132) = c_{n-1}(K_{1,m}, 132) + c_{n-2}(K_{1,m-1}, 132) \]

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<th>5</th>
<th>6</th>
<th>7</th>
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</tr>
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</table>

Proof involves crafting a simple, but nontrivial bijection

Allows us to compute \(c_n(K_{1,m}, 132)\) for any \(m, n\) quickly

- We only need the \(n = 3, 4\) and \(m = 1\) to complete the table
References


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