Solving for the Nth Roots of Unity (Numerically)

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1 Conversion into a System of Polynomials

The Nth Roots of Unity come from the roots of the complex-valued polynomial

\[ p(z) = z^n - 1 \]  

Conversion of this complex-valued, analytic function into a system of two real-valued polynomials follows by setting \( z \) equal to \( x + iy \) and separating into real and imaginary parts. For example, in the case of \( n = 2 \), we get

\[ z^2 - 1 = (x^2 - y^2 - 1) + (2ixy) \]  

which corresponds to the system of polynomial equations

\[ p(x, y) = \begin{cases} 
  x^2 - y^2 - 1 \\
  2xy 
\end{cases} \]

Note that because \( z^2 - 1 \) has two roots at \( \pm 1 \), we would expect our system to have roots at \( (\pm 1, 0) \). However, if we view our system as complex-valued instead of real-valued, we actually get four roots \( (\pm 1, 0) \) and \( (0, \pm i) \). One direct explanation is Bezout’s Theorem. Another explanation is that by setting \( z = (x + iy) \), can set \( x=0 \) and fix \( y \) accordingly, and vice versa.

In general, conversion of \( z^n - 1 \) leads to the following formula:

\[
 f(n) = \begin{cases} 
  \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^k y^{n-k} & n - k \text{ even} \\
  \sum_{k=0}^{n} (-1)^{n-k+1} \binom{n}{k} x^k y^{n-k} & n - k \text{ odd} 
\end{cases}
\]
This system has $n$ real roots and $2n - 2$ complex roots. This is because we may assume that either $x$ and $y$ are both real, or that $x$ is 0 and $y$ is imaginary, or that $y$ is 0 and $x$ is imaginary, leading to $n$ real roots and $2n$ complex roots. However, since 1 is always a root of unity, we’ve double counted it twice (since 1 is both a real root and an imaginary root) and so we subtract 2 from the total number of complex roots.