Polynomiography and Systems of Polynomials

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Vocabulary

• Monovariate: using only one variable
  • i.e. $x = 5$ is a monovariate polynomial

• Multivariate: using multiple variables
  • $xy = 5$ is a multivariable polynomial
Systems of Polynomials

\[
\begin{cases}
x^2 + y^2 - 1 = 0 \\
x - y = 0
\end{cases}
\]

This (multivariate) system has two roots! In general, it’s hard to know if a system of polynomials has a finite number of roots. It’s also quite unlikely given any arbitrary system.
How do you tell if a System has a finite number of roots?

• Given a system of polynomials $S$ over $R[x_1, x_2 \ldots x_n]$ (ring of polynomials), $S$ has a finite number of solutions iff the quotient ring

\[
\frac{R[x_1, x_2 \ldots x_n]}{S}
\]

Is finitely generated!
Methods (assuming finite roots)

• Elimination Theory
  • Elimination Theory is a generalization of Gaussian Elimination
  • Seeks to express a system of polynomial equations as a system of monovariate polynomial equations
  • The problem with this is that Elimination is computationally difficult (exponential time?)

• Newton’s Method (Generalized)
  • $x_{n+1} = x_n - Df(x_n)^{-1}f(x_n)$
  • $Df(x)^{-1}$ is the inverse of the Jacobian matrix
  • However, we can’t solve this is the Jacobian isn’t invertible. Also, calculating inverses isn’t very fun
Polynomiography for Systems of Equations

• Fix all variables except for one, thus getting a monovariate polynomial which we can then do Polynomiography on
  • Then vary the variables to get an understanding of the dynamics

• Look at the norms instead of the vector values
Bounds on Roots

• There are many known bounds on the number of roots of a single monovariate polynomial

• Generalization to systems?
What I want to do

• Develop better iterative method for solving small polynomial systems
• Polynomiography for Systems
• Find some bounds on the number of roots