Quadratic Equations over Matrices over the Quaternions

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Fields

• A field consists of a set of objects $S$ and two operations on this set.
• We will call these operations addition, denoted by $+$, and multiplication, denoted by $\ast$.
• A field must satisfy the following properties:
Additive Properties

For all $a$, $b$, and $c$ in $S$,

• (Closure) $a + b$ is in $S$

• (Identity) there exists a “0” in $S$ such that $a + 0 = a = 0 + a$.

• (Inverses) there exists a “$-a$” such that $-a + a = 0 = a + -a$.

• (Associative) $a + (b + c) = (a + b) + c$

• (Commutative) $a + b = b + a$
Multiplicative Properties

For all \( a, b, \) and \( c \) in \( S \),

- \((Closure)\) \( a \ast b \) is in \( S \)
- \((Identity)\) there exists a “1” in \( S \) such that 
  \( a \ast 1 = a = 1 \ast a \).
- \((Inverses)\) there exists a “\( a^{-1} \)” such that 
  \( a^{-1} \ast a = 0 = a \ast a^{-1} \).
- \((Associative)\) \( a \ast (b \ast c) = (a \ast b) \ast c \)
- \((Commutative)\) \( a \ast b = b \ast a \)
Distributive Property

For all \( a, b, \) and \( c \) in \( S \),

- \( a \times (b + c) = a \times b + a \times c \)
- \((b + c) \times a = b \times a + c \times a \)
Examples of Field

• Rational Numbers
• Real Numbers
• Complex Numbers
Non-examples of Fields

• Integers: (..., -3, -2, -1, 0, 1, 2, 3, ...) is not a field -- no multiplicative inverses

• The set of 2 by 2 matrices is not a field.
  – There are sometimes no multiplicative inverses. Consider
    
    \[
    \begin{bmatrix}
    1 & 0 \\
    3 & 0 \\
    \end{bmatrix}
    \]

    – Multiplication is not commutative.

\[
\begin{bmatrix}
1 & 2 \\
2 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 3 \\
3 & 0 \\
\end{bmatrix}
= 
\begin{bmatrix}
7 & 3 \\
2 & 6 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 3 \\
3 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
2 & 0 \\
\end{bmatrix}
= 
\begin{bmatrix}
7 & 2 \\
3 & 6 \\
\end{bmatrix}
\]
Division Ring

• A division ring is field that is missing one axiom.
• Multiplication in a division ring does not have to be commutative.
• That is, in a division ring $S$, there can exist $a$ and $b$ in $S$ such that $a*b$ does not equal $b*a$. 
Example: Quaternions

• The best example of a division ring is the ring of quaternions.
• Call the set of quaternions \( \mathbb{H} \).
• The quaternions can be thought of as a 4-dimensional vector space over the real numbers with basis \{1,i,j,k\}
• That is, every quaternion has the form \( x=a+bi+ cj+dk \) where \( a,b,c,d \) are real numbers.
Quaternions: Addition

- Let $x = a + bi + cj + dk$ and $y = e + fi + gj + hk$ be quaternions.
- Then $x + y = (a + e) + (b + f)i + (c + g)j + (d + h)k$.
- Clearly, addition satisfies the field properties.
Quaternions: Multiplication

• Multiplication is more complicated.
• $i, j$, and $k$ have the properties that:

\[
\begin{align*}
  i^2 & = j^2 = k^2 = -1 \\
  ij & = -ji = k \\
  jk & = -kj = i \\
  ki & = -ik = j \\
\end{align*}
\]

• Hence, multiplication is not commutative.
• However, the rest of the multiplicative field properties are satisfied.
Quaternions

• The quaternions also satisfy the distributive property.

• Thus the quaternions have all the properties of a field, except multiplication is not commutative.

• Hence, the quaternion are a division ring.
Fundamental Theorem of Algebra

- Consider the polynomial equation
  \[ f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 j \]
  where \( a_n, a_{n-1}, \ldots, a_0 \) are complex numbers and \( a_n \) does not equal 0.

- By the fundamental theorem of algebra, \( f(x) \) has \( n \) roots, including multiplicities.
Polynomial Equations over Quaternions

• This is not true over the quaternions.
• A quadratic with quaternion coefficients can have 1, 2, or infinitely many roots.
Polynomial Equations over Quaternions

- Consider \( X^2 + 1 = 0 \), where 0 and 1 are quaternions.
- A subset of solutions is \( \{ai + bj: a \text{ and } b \text{ are real, and } a^2 + b^2 = 1\} \)
- To see this, 
  \[
  (ai + bj)^2 + 1 = (ai + bj)(ai + bj) + 1 = aiai + baji + abij + bjbj + 1 = a^2i^2 + baji + abij + b^2j^2 = a^2i^2 + baji - abji + b^2j^2 = -a^2 - b^2 + 1 = -1(a^2 + b^2) + 1 = -1(1) + 1 = 0
  \]
- Thus, there are infinitely many solutions.
Equations over Matrices

• We can also consider polynomial equations over matrices.
• Let $M_2(C)$ be the set of 2 by 2 matrices, with entries that are complex numbers.
• Consider $X^2 + AX + B = 0$ over $M_2(C)$.
• It is known that the number of solutions is 0, 1, 2, 3, 4, 5, 6, or infinitely many.
Equations over Matrices: Example

Consider the equation: \( X^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \)

A subset of solutions is the set
\( \{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} : a \text{ is real} \} \)

Thus, there are infinitely many solutions.
Equation over Matrices: Example

Consider the equation: \( X^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \)

A solution has the form \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \).

Therefore, \( X^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \).

Case 1: Suppose \( a+d \) does not equal 0. Then \( b,c=0 \), so \( a^2=1 \) and \( d^2=4 \). Therefore, \( a=1,-1 \) and \( b=4,-4 \).

Case 2. Suppose \( a+d=0 \). Then \( a^2=d^2 \), so \( 1-a^2+bc=bc+d^2=4 \). This is a contradiction.

Four Solutions: \( \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 2 \end{bmatrix} \).
Equations over Matrices: Quaternions

• Let $M_2(H)$ be the set of 2 by 2 matrices with entries that are quaternions.
• Now consider $X^2+AX+B=0$ over $M_2(H)$.
• This changes the results from polynomials over $M_2(C)$.
• For example, there are now infinitely many solutions to $X^2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$.
Future Goals

• The goal of my project this is to find out in general how many solutions the polynomial $X^2 + AX + B = 0$ over $M_2(H)$ has.
Thanks

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