Sphere Packings and Number Theory: Bianchi Groups and Apollonian Circle Packings

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Bianchi Groups

Some preliminary definitions

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- Notation: Bianchi group $= SL_2(Z[\sqrt{-1}])$

Questions:
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How do we even know that such entries exist within the requirements of $SL_2(Z[\sqrt{-1}])$?

Bianchi Groups

Constructing an element

As long as the *norm* of the entries are co-prime, the remaining two entries can be constructed.
For Gaussian integers, $norm = (a + bi)(a - bi) = a^2 + b^2$.
To construct the element, we use modular arithmetic.

Bianchi Groups

Constructing an element

Example:

$$M = \begin{bmatrix} 1+2i & 8+2i \\ c & d \end{bmatrix}$$

Bianchi Groups

Constructing an element

Example:

$$M = \begin{bmatrix} \text{1+2i} & \text{8+2i} \\ c & d \end{bmatrix}$$

$$norm(1 + 2i) = 5$$

$$norm(8 + 2i) = 68$$

Bianchi Groups

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$$d = 2$$

Bianchi Groups

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Bianchi Groups

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$$(1 + 2i)c - (8 + 2i)2 = 1$$

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Bianchi Groups

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Bianchi Groups

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$$(1+2i)(5-6i)-(8+2i)2 = 1$$

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- Algebraically, a Mobius transformation on a matrix

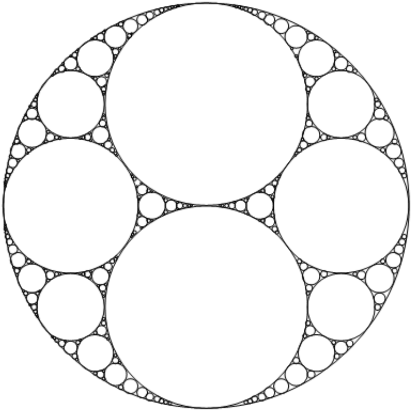
$$M = \begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$$

is $f(z) = \frac{az+b}{cz+d}$

Bianchi Groups

Mobius Transformations

- A subset of Mobius transformations on a Bianchi group turns out to give us the Apollonian circle packing!



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