Notes on Mondrony and the Developing Map

REU Summer 2025

0.1. Basic definitions

Definition 1. (Riemann surface) A **Riemann surface** is a connected, one-dimensional complex manifold. That is it is equipped with an atlas of charts that map homeomorphically to the open unit disc in $\mathbb{C}$, and the transition maps are holomorphic.

From this point on, consider a given Riemann surface X.

Definition 2. Consider $a \in X$. We write

$$\mathcal{O}_{X,a} = \mathcal{O}_p = \{\text{equivalence classes } (U,f)\}$$

where $U \ni p$ is open and $f : U \to \mathbb{C}$ is holomorphic; we decree $(U,f) = (V,g)$ if $f \equiv g$ on $W \subseteq U \cap V$ with $p \in W$, and check this is an equivalence relation. We call $\mathcal{O}_a$ the stalk at a, and the equivalence classes **germs** at a.

There is a natural evaluation mapping; we can denote a given germ f_a, where we have $f_a^k(a) = f^(k)(a)$, where (U,f) is a representative of f_a. As the derivative is local, this is independent of choice of representative.

Definition 3. (Étalé space) The **étalé space** is this context is given by $\mathcal{O} := \coprod_{a \in X} \mathcal{O}_a$. This gains a topology in the following sense; define

$$N(U,f) := \{f_z \in \mathcal{O}_z : z \in U, f_z \text{ is the germ at } z \in U \text{ given by } (U,f)\};$$

this is intuitively the set of germs at different points of U specified by (U,f). Indeed, we see each $N(U,f) \subseteq \mathcal{O}$, and so given a $f_a \in \mathcal{O}$, we say $\mathcal{N} \subseteq \mathcal{O}$ is a neighborhood (i.e. open) of f_a iff there exists a representative (U,f) of f_a such that $N(U,f) \subseteq \mathcal{N}$.

There is of course a natural projection $\pi\mathcal{O} \to X$ given $\pi(f_a) = a$.

Definition 4. (Analytic continuation along a curve) Let $f_a \in \mathcal{O}_a$. Let $\gamma : [0,1] \to X$ be a path with $\gamma(0) = a$. An **analytic continuation of** f **along** γ is a lifting $\tilde{\gamma}$ of γ to $\mathcal{O}$ w.r.t. to the projection π, such that $\tilde{\gamma}(0) = f_a$.

That is, $\tilde{\gamma} : [0,1] \to \mathcal{O}$ is a continuous map such that

$$\begin{tikzcd}
{[0,1]} \arrow[rr, "\gamma"] \arrow[dr, "\tilde{\gamma}"'] & & X \\
& \mathcal{O} \arrow[ur, "\pi"'] &
\end{tikzcd}$$

commutes and we have $\tilde{\gamma}(0) = f_a$.

Theorem 5. *(Classical monodromy) Let γ_0, γ_1 be curves in X with the same endpoints (say $a, b \in X$), oriented the same way. Let $\{\gamma_t\}_{t \in I}$ be a homotopy between γ_0 and γ_1 with the same fixed endpoints.*

Let $f_a \in \mathcal{O}_a$. Suppose that f_a can be analytically continued along all γ_t for $t \in I$. Then analytic continuation of f_a along γ_0 and along γ_1 yield the same germ at b.

Definition 6. (Fundamental group) Let D an open, connected subset of a Riemann surface X. Fix a base-point $a \in D$; then the fundamental group $\pi_1(D,a)$ is defined in the following sense:

A based loop is a continuous map $\gamma : [0,1] \to D$ with $\gamma(0) = \gamma(1) = a$. Define an equivalence relation that $\gamma_0 \cong \gamma_1$ provided there exists a homotopy between the two relative a. This becomes a group under concatenation of loops.

Definition 7. (Monodromy representation) Fix an open, connected $D \subseteq X$ and a basepoint $a \in D$. Assume we can analytically continue every $f_a \in \Omega_a$ along every loop based at a.

Define $\mathrm{Cont}_\gamma(f_a) = \tilde{\gamma}(1)$, with notation appropriated from the analytic continuation definition; i.e. it is the terminal germ given by analytic continuation.

Then there is a map $\rho : \pi_1(D, a) \to \mathrm{Aut}(\mathcal{O}_a)$ given by sending each homotopy class $[\gamma] \in \pi_1(D, a)$ to the automorphism $\rho([\gamma])$ given by $\rho([\gamma])(f_a) = \mathrm{Cont}_\gamma(f_a)$. This map ρ is known as the **monodromy representation**. It's notable that this is well-defined.

Note that here and elsewhere we will understand these automorphisms to be $\mathbb{C}$-algebra automorphisms.

Definition 8. (Multi-valued analytic functions) Fix domain D with a base-point $a \in D$. Assume every germ in $\mathcal{O}_a$ admits analytic continuation along every loop in D based at a.

It folows the monodromy representation ρ is well-defined. Fix now an initial germ $f_a \in \mathcal{O}_a$. Define the **multi-valued analytic function determined by** f_a to simply be the orbit of f_a under ρ, i.e.

$$\mathcal{M}(f_a) := \{\rho([\gamma])(f_a) : [\gamma] \in \pi_1(D, p_0)\},$$

which is *essentially* the set of germs at a given by all analytic continuations along loops based at a.

Additionally; for any $b \in D$, choose a path α from a to b. Analytically continue each $\rho([\gamma](f_a)$ along α, i.e. ranging over choices of $[\gamma]$. Collecting the resulting germs we get then in $\mathcal{O}_b$ yields the **value-set** of the multi-valued function at b, which is explicitly:

$$\{\mathrm{Cont}_\alpha(\rho([\gamma])(f_a)) : [\gamma] \in \pi_1(D, a)\} \subseteq \mathcal{O}_b.$$

We are now ready to give the definition of the developing map.

Definition 9. (Developing map) Let (S, g) a surface of constant curvature 1 with a finite set of singularities Σ. Let $S^* := S \setminus \Sigma$.

Fix a base-point $a \in S^*$. Given the curvature condition, there is a coordinate neighborhood $U_0 \ni a$ on which there is a local isometry

$$f_0 : U_0 \to \bar{C},$$

where $\bar{C}$ is the standard Riemann sphere. Of course, (U_0, f_0) specifies a germ $f_a \in \mathcal{O}_a$. Define then the multi-valued map $f : S \setminus \Sigma \to \bar{C}$ in the following way:

$$p \in S^* \longrightarrow \text{its value set} \subseteq \mathcal{O}_p \xrightarrow[\text{evaluation of}]{} \bar{C}$$

any rep at p

Indeed, as we may freely analytically continue f_a along any path not containing the singularities Σ, the monodromy representation ρ is defined.

For any point $p \in S^*$ then, choose a path α from a to p (I think the value set should be independent of choice; concatenate with a loop to resolve). This gives the value set

$$\{\mathrm{Cont}_\alpha(\rho([\gamma])(f_a)) : [\gamma] \in \pi_1(D, a)\} \subseteq \mathcal{O}_p.$$

which can be pushed to $\bar{C}$ by simply evaluating each germ in this set at p; recall these are germs of functions $S^* \to \bar{C}$. This gives you a multi-valued mapping $f : S^* \to \bar{C}$, specified by the given base-point.