Forbidden Subgraphs of Competition Graphs

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Graph Theory

• Directed graph
  ▪ Each edge has an associated direction
  ▪ Every directed, acyclic graph has at least one source and one sink node

• Subgraph
  ▪ A graph $H = (S, A)$ is a subgraph of $G = (V, E)$ if $S \subseteq V$ and all adjacency relationships of $G$ restricted to this subset are preserved in $H$

• Forbidden subgraph
  ▪ A graph $H$ is forbidden in $G$ if it is not isomorphic to any subgraph of $G$
  ▪ Forbidden subgraph characterization
    • Trees: connected graphs with no cycles
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The competition graph $C(D)$ of a digraph $D = (V,A)$ has the same vertex set as $D$, and two distinct vertices $x, y$ are adjacent in $C(D)$ if there is some vertex $z \in V$ (possibly $x$ or $y$) such that $xz, yz \in A$.
The Problem

- Let \( V \) be a finite set of \( \mathbb{R}^n \). Let \( D \) be a digraph such that \( V(D) = V \) and there is an edge from \((a_1, a_2, ..., a_n)\) to \((b_1, b_2, ..., b_n)\) if and only if \( a_i > b_i \) \( \forall i \).

- Can you list forbidden subgraphs of the competition graph \( C(D) \) of \( D \)?
Claw Graphs

- **Claw Graph**
  - A tree on $n$ nodes with one node having vertex degree $n-1$ and the other $n-1$ nodes having vertex degree 1

- **Theorem:** Any subgraph with maximum degree $n-1$ is forbidden in $C(D)$
  - **Lemma:** There is always one isolated vertex in $C(D)$
  - Let $v$ be a vertex of degree $n-1$ in $C(D)$
  - There are no loops or multiedges, so there must be an edge between $v$ and the other $n-1$ vertices, leaving no isolated vertices
Other Forbidden Subgraphs

• There are no forbidden subgraphs of $C(D)$ with maximum degree less than or equal to $n-2$
  ▪ Construction of an example
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Digraph $D$ – Note that $a_1$ is receiving arcs from all other vertices
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Competition graph $C(D)$ is a complete $K_{n-2}$ with $a_1$ isolated
Domain Restriction

- Suppose you have two lines through the origin $y = ax$, $y = bx$, where $a \neq b$, and suppose you distribute $n$ points on these two lines.

- What are the forbidden subgraphs of the competition graphs generated from our relation?
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Dominating line - A line of positive slope connecting points on two different lines
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Future Work

- Other cases in the domain restriction
- Domain restriction is currently only in two dimensions
  - Extend it to three dimensions, $k$ dimensions
- Increase the number of lines
- Consider other domain restrictions
  - Points distributed in a small grid in $\mathbb{R}^2$