Weyl's Law on Compact Heisenberg Manifolds

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Joint work with Elena Kim (MIT), Yunus E. Zeytuncu (Dearborn). Funded by the National Science Foundation (DMS-1950102 and DMS-1659203).

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where $n, m \in \mathbb{N}$.
- $N(\lambda) = \#\{(n, m) : \lambda_{n,m} \leq \lambda\}$.
- Can show:
$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda} = \frac{ab}{4\pi} = \frac{\text{vol}(\Omega)}{4\pi}.$$

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Theorem (Weyl-1911)

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- One can generalize Weyl's law to Riemannian manifolds.

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- **Example**: Dual lattice of $2\mathbb{Z} \oplus \mathbb{Z}$ is $\frac{1}{2}\mathbb{Z} \oplus \mathbb{Z}$.
- Same analysis as last time:

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- **Example**: any complex manifold.
- Every CR manifold comes with a Kohn Laplacian (CR version of standard Laplacian).

- **Goal**: Analog of Weyl's law for the Kohn Laplacian (or complex Laplacian) on spheres S^{2n-1},

$$\Box_b : L^2\left(S^{2n-1}\right) \to L^2\left(S^{2n-1}\right).$$

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- **Tool 2**: Karamata's Tauberian theorem.

Theorem (Karamata)

Let λ_j be a sequence of nonnegative numbers such that $\sum e^{-\lambda_j t}$ converges for all $t > 0$. Define $N(\lambda) = \#\{j : \lambda_j \leq \lambda\}$. For all $n > 0$ and $\alpha \in \mathbb{R}$, the following are equivalent:

(1) $\lim_{t\to 0^+} t^n \sum_{j=1}^{\infty} e^{-\lambda_j t} = \alpha$;
(2) $\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^n} = \frac{\alpha}{\Gamma(n+1)}$.

Dearborn REU 2020 Result

- Main result:

Theorem (BGS⁺2021)

Let $N(\lambda)$ be the eigenvalue counting function (with multiplicity) for $\Box_b$ on $L^2\left(S^{2n-1}\right)$. Then,

$$\lim_{n\to\infty}\frac{N\left(\lambda\right)}{\lambda^n}=\mathrm{vol}\left(S^{2n-1}\right)\frac{n-1}{n\left(2\pi\right)^n\Gamma\left(n+1\right)}\int_{-\infty}^{\infty}\left(\frac{x}{\sinh x}\right)^n e^{-(n-2)x}\,dx.$$

- Compared to Weyl's law for standard Laplacian on S^{2n-1},

$$\lim_{n\to\infty}\frac{N\left(\lambda\right)}{\lambda^{n-\frac{1}{2}}}=\frac{\mathrm{vol}\left(S^{2n-1}\right)}{2^{2n-1}\pi^{n-\frac{1}{2}}\Gamma\left(n+\frac{1}{2}\right)}.$$

The Heisenberg group

- Another model CR manifold:

Definition

The d-dimensional Heisenberg group, $\mathbb{H}_d \subseteq \mathbb{C}^{d+1}$, is the set $\mathbb{C}^d \times \mathbb{R}$ along with the group law defined by

$$(z,t) \cdot (z',t') = \left(z + z', t + t' + 2\,\mathrm{Im}\left\langle z, z'\right\rangle\right),$$

where $z, z' \in \mathbb{C}^d$; $t, t' \in \mathbb{R}$; and $\langle z, z' \rangle = z_1 \overline{z}'_1 + \cdots + z_d \overline{z}'_d$.

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- **Stanton-Tartakoff:** "Because of the group structure and its relationship to the Levi metric, analysis on the Heisenberg group is simpler than on other strongly pseudoconvex CR manifolds."

Weyl's law on $\mathbb{H}_d$?

- For $\alpha \in \mathbb{R}$ define,

$$\mathcal{L}_\alpha = -\frac{1}{2}\sum_{j=1}^{d}\left(Z_j\overline{Z}_j + \overline{Z}_j Z_j\right) + i\alpha T,$$

where

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- Every positive real number is an eigenvalue of $\mathcal{L}_\alpha$ on $L^2\left(\mathbb{H}_d\right)$.

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- **Tool 1.1** (Folland): The joint spectrum of $\mathcal{L}_0$ and $i^{-1}T$ on $L^2(M)$ is

$$\left\{\left(\frac{\pi\,|n|}{2c}(d+2j), \frac{\pi n}{2c}\right) : j \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z} \setminus \{0\}\right\} \cup \left\{\left(\frac{\pi}{2}|\xi|^2, 0\right) : \xi \in \Lambda'\right\}$$

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- **Tool 1.2** (Folland): The spectrum of $\mathcal{L}_\alpha$ on $L^2(M)$ is

$$\underbrace{\left\{ \frac{\pi |n|}{2c}(d+2j-\alpha \operatorname{sgn} n) : j \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z} \setminus \{0\} \right\}}_{\text{type } (a)} \cup \underbrace{\left\{ \frac{\pi}{2}|\xi|^2 : \xi \in \Lambda' \right\}}_{\text{type } (b)}.$$

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Analysis of $N_a(\lambda)$

- **Goal**: as $t \to 0^+$, analyze

$$t^{d+1} G(t) = t^{d+1} \sum_{i=1}^{\infty} e^{-\lambda_i t}$$

$$= t^{d+1} \sum_{\substack{n \in \mathbb{Z} \setminus \{0\} \\ j \in \mathbb{Z}_{\geq 0}}} |n|^d L \binom{j+d-1}{d-1} e^{-t \frac{\pi |n|}{2c} (d + 2j - \alpha \operatorname{sgn} n)}.$$

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- **Main idea (BGS$^+$2021)**: use the limit $t \to 0^+$ to convert the right Riemann sum $t^{d+1} G(t)$ into an integral.

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- For $\alpha < 0$, $n > 0$,

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&= t^{d+1} \sum_{n=1}^{\infty} L n^d \frac{e^{-t\frac{\pi n}{2c}(d-\alpha)}}{\left(1-e^{-t\frac{\pi n}{c}}\right)^d} \\
&= L \sum_{n=1}^{\infty} f(tn) \cdot t \ \text{ where } f(x) = x^d \frac{e^{-\frac{\pi}{2c}(d-\alpha)x}}{\left(1-e^{-\frac{\pi}{c}x}\right)^d}.
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- For $k \in \mathbb{N}$, $[t(k-1), tk]$, and $\Delta = tk - t(k-1) = t$.

Weyl's law for $\mathcal{L}_\alpha$

Theorem (FKZ2021)

Let $N(\lambda)$ be the eigenvalue counting function for $\mathcal{L}_\alpha$ on $L^2(M)$ for $-d \leq \alpha \leq d$. Then for $-d < \alpha < d$,

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d+1}} = \mathrm{vol}(M) \frac{2}{\pi^{d+1}\Gamma(d+2)} \int_{-\infty}^{\infty} \left(\frac{x}{\sinh x}\right)^d e^{-\alpha x}\, dx$$

and for $\alpha = \pm d$,

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d+1}} = \mathrm{vol}(M) \frac{2}{\pi^{d+1}\Gamma(d+2)} \frac{d}{d+1} \int_{-\infty}^{\infty} \left(\frac{x}{\sinh x}\right)^{d+1} e^{-(d-1)x}\, dx.$$

Weyl's law for $\Box_b$

Corollary (FKZ2021)

Fix $d \geq 2$. Let $N(\lambda)$ be the eigenvalue counting function for $\Box_b$ on M acting on (p,q)-forms, where $0 \leq p < d+1$, $0 < q < d$. We have that

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d+1}} = \mathrm{vol}(M) \binom{d}{p} \binom{d}{q} \frac{2}{\pi^{d+1}\Gamma(d+2)} \int_{-\infty}^{\infty} \left(\frac{x}{\sinh x}\right)^d e^{-(d-2q)x}\, dx.$$

Things to think about

- Hypersurface case: Weyl's law for the Kohn Laplacian on functions.
- Non-hypersurface case: Weyl's law the Kohn Laplacian on both functions and forms.

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