This set of notes originated from an independent study conducted in Fall 2019 under the mentorship of Purvi Gupta at Rutgers University. They will be continually updated as the author betters their own understanding of the subject. Please email colin.fan@rutgers.edu regarding any questions or errors. Last updated: August 25, 2021.

All mistakes and misunderstandings, especially in terms of interpretation are the author's alone. The main reference is Ahlfors's *Conformal Invariants*. The secondary reference is Beardon and Minda's *The Hyperbolic Metric and Geometric Function Theory*. The author thanks Purvi Gupta for all her mentorship beyond the aforementioned independent study.

Contents

Facts, notation, and some motivation **2**

The hyperbolic metric on $\mathbb{D}$ **3**

The Schwarz-Pick theorem **6**

Boundary behavior **8**
- Julia's lemma . . . 8
- Angular derivatives . . . 10
- Löwner's lemma . . . 14

Ahlfors's generalization of Schwarz-Pick **15**

Applications of Ahlfors's lemma **18**
- Bloch's theorem . . . 18
- Bounds for Poincaré metrics . . . 19
- The Picard theorems . . . 22

Appendix **26**
- Nevanlinna-Pick interpolation . . . 26
- Two lemmas for Bloch's theorem . . . 28

Facts, notation, and some motivation

The unit disk and upper half-plane in $\mathbb{C}$ are denoted by $\mathbb{D}$ and $\mathbb{H}$ respectively.

Let U be an open set in $\mathbb{C}$. The (holomorphic) automorphisms of U are denoted by $\mathcal{A}(U)$. One can show that,

$$\mathcal{A}(\mathbb{D}) = \left\{ e^{i\theta} \frac{z-\alpha}{1-\overline{\alpha}z} : \theta \in \mathbb{R}, \alpha \in \mathbb{D} \right\}.$$

We say that a property or function on U is a *conformal invariant* if it is preserved under composition with elements in $\mathcal{A}(U)$.

Let X be a set and (Y,d) a metric space. One can give X a pseudometric structure via the following: given any $f : X \to Y$, define $\delta : X \times X \to \mathbb{R}$ by the rule

$$\delta(x_1, x_2) = d(f(x_1), f(x_2)).$$

This essentially copies the metric structure of Y onto X, except possibly definiteness. Definiteness is preserved when f is injective, in which case the structure is copied exactly. We say that δ is the *pullback* of the metric d under f.

Similar to above, we can consider pullbacks of Riemannian metrics. Fix M to be a smooth manifold and (N,g) a Riemannian manifold. If $F : M \to N$ is smooth, we can give X a Riemannian pseudometric structure by defining $F^*g : C^\infty(TM) \otimes C^\infty(TM) \to C^\infty(M)$ as

$$(F^*g)(v,w)(p) = g((dF_p(v), dF_p(w)))(F(p)).$$

Similar to the case for metric spaces, we copy the Riemmanian geometry of N onto M, except possibly definiteness. Definiteness is preserved if and only if F is a smooth immersion. We say that F^*g is the *pullback* of the metric g under F.

The hyperbolic metric on $\mathbb{D}$

Let $\delta : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$ be defined by the rule,

$$\delta(z, w) := \left| \frac{z - w}{1 - \overline{w}z} \right|.$$

We refer to δ as the *Ahlfors metric*.

Theorem 1. *The Ahlfors metric is a conformal invariant. That is, for any $S \in \mathcal{A}(\mathbb{D})$,*

$$\delta(S(z), S(w)) = \delta(z, w).$$

This fact follows by expanding the left-hand side in terms of z and w. Moreover, it follows immediately that $\delta(z, w) < 1$. Alternatively, one can use the identity,

$$1 - \delta(z, w)^2 = \frac{\left(1 - |z|^2\right)\left(1 - |w|^2\right)}{|1 - \overline{w}z|}.$$

This identity will be used shortly.

Example 2. Disks in the Ahlfors metric.

Fix $w = a + ib \in \mathbb{D}$. It follows that the Ahlfors-ball centered at w is defined by $\{z \in \mathbb{C} : \delta(z, w) < R\}$. This is equivalent to

$$\left\{ z \in \mathbb{C} : \left| z - w\frac{1 - R^2}{1 - R^2 |w|^2} \right| < R\frac{1 - |w|^2}{1 - R^2 |w|^2} \right\}.$$

That is, Ahlfors-balls are identical to standard Euclidean balls in the disk with an offset center. Note that the Ahlfors center, Euclidean center and the origin are colinear, and the Euclidean center is closer to the origin due to the $\frac{1-R^2}{1-R^2|w|^2}$ factor. Moreover, we have that $\mathbb{D}$ under the Ahlfors metric is topologically equivalent to $\mathbb{D}$ under the Euclidean metric.

The Ahlfors metric is a metric in the analytical sense, which will be shown later. However, this tells us this is not the metric we will want to equip $\mathbb{D}$ with, as we require a metric in the geometric sense.

If we take $w \to z$, we have that

$$\frac{|dS(z)|}{1 - |S(z)|^2} = \frac{|dz|}{1 - |z|^2}.$$

Thus, the Riemannian metric $g : C^\infty(T\mathbb{D}) \otimes C^\infty(T\mathbb{D}) \to C^\infty(\mathbb{D})$ defined by the rule,

$$g(X, Y)(p) = 4\frac{\langle X_p, Y_p \rangle}{\left(1 - |p|^2\right)^2}$$

is a conformal invariant. That is, if $S \in \mathcal{A}(\mathbb{D})$ is defined by $S(z) = e^{i\theta}\frac{z-\alpha}{1-\overline{\alpha}z}$, we have that

$$\begin{aligned} g\left((DS)_p X, (DS)_p Y\right)(S(p)) &= \frac{4}{\left(1 - |S(p)|^2\right)^2} \left\langle e^{i\theta}\frac{1 - |\alpha|^2}{(1 - \overline{\alpha}z)^2} X_p, e^{i\theta}\frac{1 - |\alpha|^2}{(1 - \overline{\alpha}z)^2} Y_p \right\rangle \\ &= 4\left(\frac{|1 - \overline{\alpha}p|^2}{\left(1 - |p|^2\right)\left(1 - |\alpha|^2\right)}\right)^2 \cdot \left(\frac{1 - |\alpha|^2}{|1 - \overline{\alpha}z|^2}\right)^2 \langle X_p, Y_p \rangle \\ &= g(X, Y)(p). \end{aligned}$$

So, the element of length for g,

$$ds = \frac{2\,|dz|}{1-|z|^2}$$

is also a conformal invariant. We will refer to this element of length on $\mathbb{D}$ as the *hyperbolic* or *Poincaré metric*. When equipping $\mathbb{D}$ with the hyperbolic metric, we call it the *Poincaré disk*. Moreover the hyperbolic metric will be denoted as, $\lambda\,|dz|$ where

$$\lambda(z) \coloneqq \frac{2}{1-|z|^2}.$$

It follows that the hyperbolic length of rectifiable arcs,

$$L(\gamma) = \int_\gamma ds = \int_\gamma \frac{2\,|dz|}{1-|z|^2}$$

is conformally invariant, as

$$L(S(\gamma)) = \int_{S(\gamma)} \frac{2\,|dz|}{1-|z|^2} = \int_\gamma \frac{2\,|dS(z)|}{1-|S(z)|^2} = \int_\gamma \frac{2\,|dz|}{1-|z|^2} = L(\gamma).$$

We equip $\mathbb{D}$ with the hyperbolic geometry, induced by the hyperbolic metric. We can then induce a metric d in the analysis sense, called the *hyperbolic distance* by

$$d(z,w) \coloneqq \inf_\gamma L(\gamma),$$

where the infimum is taken over all continuously differentiable curves joining z to w in $\mathbb{D}$.

Theorem 3. *The hyperbolic distance is a conformal invariant.*

Proof. Recall that L is a conformal invaraint. This implies,

$$\{L(\gamma) : \gamma \text{ connects } z \text{ and } w\} = \{L(S(\gamma)) : \gamma \text{ connects } S(z) \text{ and } S(w)\} \subseteq \{L(\gamma) : \gamma \text{ connects } S(z) \text{ and } S(w)\}.$$

So, $d(S(z), S(w)) \leq d(z,w)$. Since $S^{-1} \in \mathcal{A}(\mathbb{D})$, by the same line of reasoning as above, $d\left(S^{-1}(z), S^{-1}(w)\right) \leq d(z,w)$, which implies $d(z,w) \leq d(S(z), S(w))$. ∎

Since the hyperbolic disk has a way of measuring distances between two points, it is a natural to ask what the geodesics are. Computationally, it is easier to look at $\mathbb{H}$.

Recall that $\mathbb{H}$ and $\mathbb{D}$ are conformally equivalent via $f : \mathbb{H} \to \mathbb{D}$ defined by the rule

$$f(z) = \frac{z-i}{z+i}.$$

If we equip $\mathbb{H}$ with the metric that has element of length, $ds = |dz|/y$, then f is an isometry[1]. So it suffices to compute geodesics in $\mathbb{H}$. We start by looking at the simplest case: geodesics from i to ir with $r > 1$.

Theorem 4. *Fix $r \in (1,\infty)$. Then the geodesic from i to r is a Euclidean straight line.*

Proof. Let $\gamma(t) = i(1-t) + irt$ for $t \in [0,1]$. Then,

$$d(i, ir) \leq L(\gamma) = \int_0^1 \frac{\sqrt{(-1+r)^2}}{1-t+rt}\,dt = \int_1^r \frac{1}{t}\,dt = \log r.$$

Now fix $\mu : [0,1] \to \mathbb{H}$ continuously differentiable function connecting i and r with real part x and imaginary part y. It follows that,

$$L(\mu) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\,dt \geq \int_0^1 \frac{|y'(t)|}{y(t)}\,dt \geq \int_0^1 \frac{y'(t)}{y(t)} = \log y(1) - \log y(0) = \log r.$$

Note that the first equality holds above if and only if $x'(t) = 0$ everywhere as x' is continuous, and similarly the second equality holds if and only if $y'(t) \geq 0$ everywhere. So, the shortest path between i and ir is γ. ∎

[1] This definition can be seen as natural, after looking at the pullback of f under the hyperbolic metric.

Note that $\mathcal{A}(\mathbb{H}) = \left\{ \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$. Moreover, the hyperbolic metric for the half-plane is conformally invariant under $\mathcal{A}(\mathbb{H})$, and therefore sends geodesics to geodesics. Note that the line connecting i and ir forms a right angle with the real axis. Since Mobiüs transforms are conformal, and send circles and lines to circles and lines, we have that the geodesics of $\mathbb{H}$ are segments of straight lines or arcs of circles that intersect the real axis at a right angle.

Under f, we see that the real axis and a point at infinity is sent to the boundary of $\mathbb{D}$ and 1. Since f itself is a Mobiüs transform, the geodesics of $\mathbb{D}$ must be radial lines, cutting the boundary at right angles, and orthogonal circles to the boundary. Specifically, if z and w are colinear with the origin, then the geodesic connecting them is a Euclidean straight line. If not, then they are connected by an arc of a circle.

Example 5. Horocycles A *horocycle* in hyperbolic geometry is a curve where every geodesic that intersects it tranversally all converge asymptotically in the same direction. Examples of horocycles in the Poincaré disk are Euclidean circles contained in $\mathbb{D}$ that are tangent to any boundary point. Explicitly, for fixed $\theta \in \mathbb{R}$ with $|\theta| = 1$, and $R > 0$,

$$\left\{ z \in \mathbb{D} : \frac{|1-z|^2}{1-|z|^2} = R \right\}$$

is a horocycle. As a Euclidean circle it is given by the equation,

$$\left(x - \frac{1}{R+1} \right)^2 + y^2 = \frac{R^2}{(R+1)^2}.$$

We will often abuse language and say the interior of the horocycle is a horocycle. In this case, these horocycles are hyperbolic disks that are in some sense "centered at infinity." This centering at infinity will be justified in the section of Julia's lemma.

Example 6. The distance from 0 to any $z \in \mathbb{D}$ is $2 \operatorname{arctanh}(|z|)$.

Proof. Let $\gamma = zt$ for $t \in [0, 1]$. It follows that,

$$L(\gamma) = \int_0^1 \frac{|z|}{1 - |z|^2 t^2} \, dt = \log \left(\frac{1+|z|}{1-|z|} \right) = 2 \operatorname{arctanh}(|z|).$$

■

Example 7. For any $z, w \in \mathbb{D}$, we have that $d(z, w) = 2 \operatorname{arctanh}(\delta(z, w))$.

Proof. Consider $T(\zeta) = \frac{\zeta - w}{1 - \overline{w}\zeta} \in \mathcal{A}(\mathbb{D})$. Then,

$$d(z, w) = d(T(z), T(w)) = d(T(z), 0) = 2 \operatorname{arctanh}(|T(z)|) = 2 \operatorname{arctanh}(\delta(z, w)).$$

■

This fact implies that the topology induced by the hyperbolic distance is equivalent to that of the Ahlfors metric, and therefore is equivalent to the Euclidean topology

Theorem 8. *The Ahlfors metric is a metric (in the analysis sense).*

Proof. Note that if d is a metric, and $g : [0, \infty) \to [0, \infty)$ is injective, and concave, then $g \circ d$ is a metric. ■

The Schwarz-Pick theorem

Recall the Schwarz lemma,

Theorem 9. *Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic. If $f(0) = 0$ and $|f(z)| < 1$ for all $z \in \mathbb{D}$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Moreover, if there exists a non-zero $z_0 \in \mathbb{D}$ such that $|f(z_0)| = |z_0|$, or $|f'(0)| = 1$, then $f(z) = e^{i\theta} z$ where θ is real.*

One needs not require a fixed point, and with this slight generalization, the Schwarz lemma takes on an invariant form: the Schwarz-Pick theorem.

Theorem 10. *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Then f reduces or preserves distance with respect to the Ahlfors metric and hyperbolic lengths of arcs. That is,*

$$\left| \frac{f(z) - f(w)}{1 - \overline{f(w)} f(z)} \right| \leq \left| \frac{z - w}{1 - \overline{w} z} \right| \text{ and } \frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Moreover, the two inequalities are equivalent, and if there exists $z_0 \in \mathbb{D}$ such that equality holds in either inequality, then $f \in \mathcal{A}(\mathbb{D})$.

Proof. We first show equivalence. The first inequality implies the second via taking $w \to z$. To see the converse, consider any $z, w \in \mathbb{D}$ and γ a path connecting these two points. It follows that $L(f \circ \gamma) \leq L(\gamma)$. This implies $d(f(z), f(w)) \leq d(z, w)$. Since tanh is increasing, it follows that $\delta(f(z), f(w)) \leq \delta(z, w)$.

Fix $w \in \mathbb{D}$. Let $g_w, h_w : \mathbb{D} \to \mathbb{D}$ be defined by the rules,

$$g_w(z) = \frac{z - f(w)}{1 - \overline{f(w)} z} \text{ and } h_w(z) = \frac{z - w}{\overline{w} z - 1}.$$

Since $g_w \circ f \circ h_w$ has fixed point 0, by the Schwarz lemma,

$$|g_w(f(h_w(z)))| \leq |z|.$$

Then since $h_w = h_w^{-1}$,

$$|g_w(f(z))| \leq \left| h_w^{-1}(z) \right|.$$

That is, $\delta(f(z), f(w)) \leq \delta(z, w)$.

Now suppose there exists $\alpha, \beta \in \mathbb{D}$ such that $\delta(f(\alpha), f(\beta)) = \delta(\alpha, \beta)$. It follows that

$$|g_\beta(f(h_\beta(\alpha)))| = |\alpha|.$$

So, $g_\beta \circ f \circ h_\beta$ is a rotation. Since $g_\beta, h_\beta \in \mathcal{A}(\mathbb{D})$, it follows that $f \in \mathcal{A}(\mathbb{D})$. ■

Consider a hyperbolic disk, $B = \{z \in \mathbb{D} : |(z - w) / (1 - \overline{w} z)| < r\}$ for some fixed $w \in \mathbb{D}$ and $r > 0$. For $f : \mathbb{D} \to \mathbb{D}$ holomorphic, geometrically, the Schwarz-Pick theorem says that $f(B)$ is contained in the hyperbolic disk of radius r centered at $f(w)$.

The Schwarz-Pick theorem characterizes the isometry group of $\mathbb{D}$ as $\mathcal{A}(\mathbb{D})$. Moreover, our objects of most importance: holomorphic functions, are all contractions with respect to the hyperbolic disk, which is untrue in Euclidean geometry. This supports the viewpoint that the natural and "correct" geometry for complex analysis to take place is within hyperbolic geometry.

Example 11. If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, then $|f'(0)| \leq 1$. Moreover, $|f'(0)| = 1$ if and only if $f(z) = e^{i\theta} z$ where θ is real.

Proof. By Schwarz-Pick, $|f'(0)| \leq 1 - |f(0)|^2 \leq 1$. If $|f'(0)| = 1$, then $f(0) = 0$. ■

Example 12. Schwarz-Pick theorem for the half-plane model.

If $f : \mathbb{H} \to \mathbb{H}$ is holomorphic, then for all $z, w \in \mathbb{H}$,

$$\left| \frac{f(z) - f(w)}{f(z) - \overline{f(w)}} \right| \leq \left| \frac{z - w}{z - \overline{w}} \right|.$$

Proof. Consider $T : \mathbb{H} \to \mathbb{D}$ defined by the rule,

$$T(z) \coloneqq \frac{z - i}{z + i}.$$

It follows that $g \coloneqq T \circ f \circ T^{-1}$ is an endomorphism on $\mathbb{D}$. By Schwarz-Pick,

$$\delta(g(z), g(w)) \leq \delta(z, w).$$

This implies,

$$\delta(T(f(z)), T(f(w))) \leq \delta(T(z), T(w)).$$

It follows that,

$$\begin{aligned} \delta(T(z), T(w)) &= \left| \frac{\frac{z-i}{z+i} - \frac{w-i}{w+i}}{1 - \overline{\left(\frac{w-i}{w+i}\right)} \cdot \frac{z-i}{z+i}} \right| \\ &= \left| \frac{(w+i)\overline{(w+i)}(z-i) - (w-i)\overline{(w+i)}(z+i)}{(w+i)\overline{(w+i)}(z+i) - (w+i)\overline{(w-i)}(z-i)} \right| \\ &= \left| \frac{z - w}{z - \overline{w}} \right|. \end{aligned}$$

The computation for the left-hand side is identical. ∎

Boundary behavior

We give two applications of Schwarz-Pick to describe the behavior of holomorphic functions on the boundary: Julia's lemma, and Löwner's lemma.

Julia's lemma

Recall that the Schwarz-Pick theorem says that the image of a hyperbolic disk of center w and radius r under any $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ is contained in the hyperbolic disk centered at $f(w)$ and radius r. What can be said about the boundary behavior of f? Does the same phenomenon occur for "disks centered at infinity?" That is, what can we say about the image of horocycles under f? For simplicity, we consider the behavior at $1 \in \partial\mathbb{D}$.

Let $\{z_n\} \subseteq \mathbb{D}$ be a sequence that converges to 1, and $\{R_n\} \subseteq \mathbb{R}_{>0}$ so that

$$\frac{1-|z_n|}{1-R_n} \to k \neq 0, \infty.$$

Note that this says $\{R_n\}$ converges to 1, but does not converge faster than $|z_n|$. Let K_n be the hyperbolic disks of center z_n and radius R_n.

Theorem 13. *The K_n tend to the horocycle K_∞ defined by,*

$$\left\{ z \in \mathbb{D} : \frac{|1-z|^2}{1-|z|^2} < k \right\}.$$

The convergence $K_n \to K_\infty$ means that if $z \in K_n$ for infinitely many n, then $z \in \overline{K}_\infty$, the closure of K_∞. Moreover, if $z \in K_\infty$, then $z \in K_n$ for all sufficiently large n.

Proof. By definition, $z \in K_n$ if and only if $\delta(z, z_n) < R_n$. That is, if and only if

$$1 - R_n^2 < 1 - \delta(z, z_n)^2 = \frac{\left(1-|z_n|^2\right)\left(1-|z|^2\right)}{|1-\overline{z_n}z|^2}.$$

Thus, $z \in K_n$ if and only if

$$\frac{|1-\overline{z}_n z|^2}{1-|z|^2} < \frac{1-|z_n|^2}{1-R_n^2}.$$

If $z \in K_n$ for infinitely many n, by passing to a subsequence, and taking the limit, we have that $z \in \overline{K}_\infty$. Conversely, if $z \in K_\infty$, then both

$$\lim_{n\to\infty} \frac{|1-\overline{z}_n z|^2}{1-|z|^2} = \frac{|1-z|^2}{1-|z|^2} < k.$$

Then since,

$$\lim_{n\to\infty} \frac{1-|z_n|^2}{1-R_n^2} = k,$$

it follows that for sufficiently large n, $z \in K_n$. ■

This justifies the intuition that horocycles are hyperbolic disks centered at infinity. We now state Julia's lemma.

Theorem 14 (Julia's lemma)**.** *Fix $k > 0$, and $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$. If there exists a sequence $\{z_n\} \subseteq \mathbb{D}$ so that $z_n \to 1$, $f(z_n) \to 1$ and*

$$\frac{1-|f(z_n)|}{1-|z_n|} \to \alpha \neq \infty,$$

then

$$\frac{|1-z|^2}{1-|z|^2} \leq k \text{ implies } \frac{|1-f(z)|^2}{1-|f(z)|^2} \leq \alpha k.$$

That is, if $z \in \overline{K}_\infty$, then $f(K_\infty) \subseteq \overline{K}'_\infty$ where K'_∞ is the horocycle centered at 1 of radius αk.

Proof. Fix N sufficiently large so that $1-|z_n|<k$ for all $n \geq N$. Choose R_n so that $(1-|z_n|)/(1-R_n)=k$ for $n \geq N$. Note that this implies $0<R_n<1$. Let $K_n = K(z_n, R_n)$. By the Schwarz-Pick theorem, $f(K_n) \subseteq K_n'$ where K_n' is the hyperbolic disk of radius R_n and center $f(z_n)$. By the previous theorem, the K_n converge to the horocycle K_∞ of radius k. It follows that,

$$\frac{1-|f(z_n)|}{1-R_n} = \frac{1-|f(z_n)|}{1-|z_n|} \cdot \frac{1-|z_n|}{1-R_n} \to \alpha k.$$

Now fix $z \in K_\infty$. By the previous theorem, $z \in K_n$ for infinitely many n. Thus, $f(z) \in K_n'$ for infinitely many n, which implies $f(z) \in \overline{K}'_\infty$. ■

Remark 15. The assumption that $f(z_n) \to 1$ is not necessary. Neither is the assumption that $|f(z_n)| \to 1$. The reason it is assumed in the theorem is to simplify the computations as we only computed horocycles centered at 1. Moreover, this α-distortion term can be minimized by taking the limit infimum as $z \to 1$ of $(1-|f(z)|)/(1-|z|)$.

Since k was arbitrary, we obtain this boundary form of the Schwarz-Pick theorem.

Corollary 16. For all $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$, satisfying the conditions in Julia's lemma, and $z \in \mathbb{D}$,

$$\frac{|1-f(z)|^2}{1-|f(z)|^2} \leq \alpha \frac{|1-z|^2}{1-|z|^2}.$$

Equivalently,

$$\beta = \sup\left(\frac{|1-f(z)|^2}{1-|f(z)|^2} \cdot \frac{1-|z|^2}{|1-z|^2}\right) \leq \alpha.$$

Note that β is nonzero as f is not identically 1, and if $\beta = \infty$, then there is no finite α for which the difference quotient $(1-|f(z_n)|)/(1-|z_n|)$ can limit to.

Angular derivatives

Julia's lemma allows us to study the derivative of $f \in \mathcal{O}(\mathbb{D},\mathbb{D})$ at the boundary. It makes sense to talk about the quotient $(1-f(z))/(1-z)$, and $f'(z)$ as $z \to 1$. We look to answer the following questions: when do these limit exists, and if they exist, when are they equal. Let us first consider a radial limit.

Theorem 17. *Fix $f \in \mathcal{O}(\mathbb{D},\mathbb{D})$. For $x \in \mathbb{D} \cap \mathbb{R}$,*

$$\lim_{x\to 1} \frac{1-f(x)}{1-x} = \beta$$

where

$$\beta = \sup\left(\frac{|1-f(z)|^2}{1-|f(z)|^2}\cdot\frac{1-|z|^2}{|1-z|^2}\right).$$

Proof. First fix $\beta < \infty$. Let $\{x_n\}$ be a real sequence converging to 1. Since,

$$\frac{1-|f(x_n)|^2}{1-\left|f(x_n)^2\right|}\cdot\frac{1-|x_n|^2}{|1-x_n|^2} \le \beta,$$

we have that

$$|1-f(x_n)|^2 \le \frac{|1-f(x_n)|^2}{1-|f(x_n)|^2} \le \beta\frac{(1-x_n)^2}{(1-x_n)(1+x_n)} = \beta\frac{1-x_n}{1+x_n}.$$

Thus, $f(x_n) \to 1$ automatically. Similarly,

$$\beta \ge \frac{|1-f(x_n)|^2}{1-|f(x_n)|^2}\cdot\frac{1-x_n^2}{|1-x_n|^2} \ge \frac{|1-f(x_n)|^2}{1-|f(x_n)|^2}\cdot\frac{1+x_n}{1-x_n} \ge \frac{|1-f(x_n)|}{1+|f(x_n)|}\cdot\frac{1+x_n}{1-x_n} \ge \frac{1-|f(x_n)|}{1-x_n}\cdot\frac{1+x_n}{1+|f(x_n)|}.$$

Taking the limit, we see that

$$\lim_{n\to\infty}\frac{1-|f(x_n)|}{1-x_n} \le \beta.$$

Since x_n was arbitrary, it follows that the distortion factor, α along the x-axis is exactly β. Moreover, by the above chain of inequalities,

$$\lim_{x\to 1}\frac{1-|f(x)|}{1-x} = \lim_{x\to 1}\frac{|1-f(x)|}{1-x} = \beta.$$

Note that since $\beta \neq 0, \infty$, it follows that

$$\lim_{x\to 1}\frac{|1-f(x)|}{1-|f(x)|} = 1.$$

In particular, for all $M > 1$, there exists a neighborhood of 1 so that for x in that neighborhood,

$$\frac{|1-f(x)|}{1-|f(x)|} < M.$$

This implies for x close to 1, the argument of $f(x)$ is close to 0 (Stolz angle). Since $1-z$ is a rotation about $1/2$, it follows that $\arg(1-f(x)) \to 0$ as $x \to 1$. This implies,

$$\lim_{x\to 1}\frac{1-f(x)}{1-x} = \lim_{x\to 1}\left|\frac{1-f(x)}{1-x}\right| = \beta.$$

For the case that $\beta = \infty$, note that

$$\lim_{x\to 1}\frac{1-|f(x)|}{1-x} = \infty.$$

Monotonicity implies,

$$\lim_{x\to 1}\frac{1-|f(x)|}{1-x} = \lim_{x\to 1}\frac{|1-f(x)|}{1-x} = \infty,$$

and therefore $\lim_{x\to 1}(1-f(x))/(1-x) = \infty$. ∎

Thus, the difference quotient $(1 - f(z)) / (1 - z)$ always has a radial limit. We can do better: if $z \to 1$ so that there exists $M > 1$ with $|1 - z| \le M(1 - |z|)$ (Stolz angle), then the difference quotient limit exists. This condition means that z approaches 1 within an angle less than π, and the limit is referred to as an angular limit.

Theorem 18. *Fix $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$. The quotient,*

$$\frac{1 - f(z)}{1 - z}$$

always has an angular limit as $z \to 1$. Explicitly, the limit is

$$\beta = \sup \left(\frac{|1 - f(z)|^2}{1 - |f(z)|^2} \cdot \frac{1 - |z|^2}{|1 - z|^2} \right),$$

and therefore is either ∞ or a positive real number. If it is finite, $f'(z)$ has the same angular limit.

Proof. We first need two computational results. The first: for $z \neq 1$,

$$\operatorname{Re} \frac{1 + z}{1 - z} = \frac{1 - |z|^2}{|1 - z|^2}.$$

The second: for any non-constant $g \in \mathcal{O}(\mathbb{D})$ satisfying $\operatorname{Re} g \ge 0$, there exists $F \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ so that

$$g = \frac{1 + F}{1 - F}.$$

Note in fact that $\operatorname{Re} g > 0$ by the open mapping theorem.

First assume $\beta = \infty$. For angular approach, this implies,

$$\lim_{z \to 1} \frac{1 - |f(z)|}{1 - |z|} = \infty.$$

Since $|1 - z| \le M(1 - |z|)$, by the triangle inequality,

$$\lim_{z \to 1} \left| \frac{1 - f(z)}{1 - z} \right| = \infty.$$

Thus,

$$\lim_{z \to 1} \frac{1 - f(z)}{1 - z} = \beta.$$

Now assume $\beta < \infty$. We reduce to the $\beta = \infty$ case. By the first fact, for all $z \in \mathbb{D}$,

$$0 \le \beta \operatorname{Re} \frac{1 + f(z)}{1 - f(z)} - \operatorname{Re} \frac{1 + z}{1 - z}.$$

By the second fact, there exists $F \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ so that

$$\beta \frac{1 + f(z)}{1 - f(z)} - \frac{1 + z}{1 - z} = \frac{1 + F(z)}{1 - F(z)}.$$

Let

$$\beta' = \sup \left(\frac{|1 - F(z)|^2}{1 - |F(z)|^2} \cdot \frac{1 - |z|^2}{|1 - z|^2} \right).$$

If $\beta' < \infty$, then

$$\operatorname{Re} \frac{1 + z}{1 - z} \le \beta' \operatorname{Re} \frac{1 + F(z)}{1 - F(z)} = \beta' \left(\beta \operatorname{Re} \frac{1 + f(z)}{1 - f(z)} - \operatorname{Re} \frac{1 + z}{1 - z} \right).$$

That is,

$$\left(1+\frac{1}{\beta'}\right)\operatorname{Re}\frac{1+z}{1-z}\leq\beta\operatorname{Re}\frac{1+f(z)}{1-f(z)}.$$

This contradicts the definition of β, and therefore $\beta'=\infty$. In particular, for angular approach,

$$\lim_{z\to1}\frac{1-z}{1-F(z)}=0.$$

Since

$$\beta\frac{1+f(z)}{1-f(z)}(1-z)-(1+z)=\frac{1+F(z)}{1-F(z)}(1-z),$$

in the angular limit,

$$\lim_{z\to1}\frac{1-z}{1-f(z)}=\frac{1}{\beta}.$$

We now show that finite β, in the angular limit, $f'(z)\to\beta$. By differentiation,

$$\beta f'(z)(1-f(z))^{-2}-(1-z)^{-2}=F'(z)(1-F(z))^{-2}.$$

By the Schwarz-Pick theorem, and the fact that $|1-z|\leq M(1-|z|)$,

$$\begin{aligned}\left|\beta f'(z)\frac{(1-z)^2}{(1-f(z))^2}-1\right|&=|F'(z)|\left|\frac{1-z}{1-F(z)}\right|^2\\&\leq\left|\frac{1-z}{1-F(z)}\right|^2\frac{1-|F(z)|^2}{1-|z|^2}\\&\leq M^2\frac{(1-|z|)^2}{|1-F(z)|^2}\frac{1-|F(z)|^2}{1-|z|^2}\\&\leq M^2\frac{1-|z|}{|1-F(z)|^2}\left(1-|F(z)|^2\right)\\&\leq 2M^2\frac{1-|z|}{1-|F(z)|}.\end{aligned}$$

It follows that $f'(z)\to\beta$. ■

When $\beta\neq\infty$, we call it the angular derivative for f at 1. In this case, we have that $f(z)\to1$ as $z\to1$ in an angle. So, β is the angular limit of the quotient, $(f(z)-f(1))/(z-1)$, which is equal to the angular limit of $f'(z)$. Also, recalling that $\beta>0$, we note that provided everything stays non-tangent to 1 in the disk, f is conformal at 1.

Now note that $1\in\partial\mathbb{D}$ was not special. Given a function $f_1(z)\to e^{i\delta}$ as $z\to e^{i\gamma}$, we can consider

$$f(z)=e^{-i\delta}f_1\left(e^{-i\gamma}z\right),$$

which would transfer to the setting given.

We can also consider angular derivatives in the half-plane, which is again, computationally more convenient than the disk.

Corollary 19. Suppose $f=u+iv$ maps the right half-plane into itself. It follows that,

$$\lim_{z\to\infty}\frac{f(z)}{z}=\lim_{z\to\infty}\frac{u(z)}{x}=c=\inf\frac{u(z)}{x}$$

so long as our limits are restricted to $|\arg z|\leq\pi/2-\varepsilon$ for some fixed $\varepsilon>0$.

Proof. We convert the map of half-planes to that of disks. Consider $g : \mathbb{D} \to \mathbb{D}$ defined by the rule,

$$g\left(w\right) = \frac{f\left(z\right) - 1}{f\left(z\right) + 1}$$

where z is in the right half-plane so that $\left(z-1\right)/\left(z+1\right) = w$. Note that the region given by the Stolz angle, $|1 - z| \leq M\left(1 - |z|\right)$ corresponds exactly to the region given by $|\arg z| \leq \pi/2 - \varepsilon$ for some fixed ε dependent on M.

Define,

$$\beta = \sup\left(\frac{\left|1 - g\left(w\right)\right|^2}{1 - \left|g\left(w\right)\right|^2} \cdot \frac{1 - |w|^2}{|1 - w|^2}\right).$$

Now recall the first computational result from before. It follows that,

$$\beta = \sup\left(\left(\operatorname{Re}\frac{1 + g\left(w\right)}{1 - g\left(w\right)}\right)^{-1} \operatorname{Re}\frac{1 + w}{1 - w}\right) = \sup\frac{\operatorname{Re} z}{u\left(z\right)}.$$

Now define $c = \frac{1}{\beta} = \inf u\left(z\right)/x$. Since $\beta > 0$, we have that $c \geq 0$. It follows that in the angular limit for w,

$$c = \lim_{w\to 1}\frac{1 - w}{1 - g\left(w\right)} = \lim_{z\to\infty}\frac{1 - \frac{z-1}{z+1}}{1 - \frac{f(z)-1}{f(z)+1}} = \lim_{z\to\infty}\frac{1 + f\left(z\right)}{1 + z}.$$

This implies $f/z \to c$ as $z \to \infty$ within $|\arg z| \leq \pi/2 - \varepsilon$. For the last claimed equality we see that in an angle,

$$\lim_{z\to\infty}\frac{u\left(z\right)}{x} = \lim_{w\to 1}\frac{1 - \left|g\left(w\right)\right|^2}{\left|1 - g\left(w\right)\right|^2} \cdot \frac{|1 - w|^2}{1 - |w|^2} = \lim_{w\to 1}\frac{1}{\beta^2}\frac{1 - \left|g\left(w\right)\right|^2}{1 - |w|^2} = \lim_{w\to 1}\frac{1}{\beta^2}\frac{1 - \left|g\left(w\right)\right|}{1 - |w|}\frac{1 + \left|g\left(w\right)\right|}{1 + |w|} = c.$$

■

Löwner's lemma

What can be said about how $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ acts directly on the boundary, assuming such an action makes sense? Suppose we have the property that $|f(z)| \to 1$ as z approaches an open arc $\gamma \subseteq \partial\mathbb{D}$. By the reflection principle, f extends analytically to γ. Moreover, $f'(\zeta) \neq 0$ for $\zeta \in \gamma$ as $|f(z)|$ is increasing as $z \to \zeta$. Moreover, this implies as $\arg\zeta$ increases, so does $\arg f(\zeta)$ as $f'(\zeta) \neq 0$ and holomorphic functions preserve orientation.

We now state Löwner's lemma, which will follow directly from angular derivatives. Intuitively, it states that if we contract the unit disk while maintaining a self-map on part of the boundary, then we must have stretched the boundary.

Theorem 20. *Fix $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ and $\gamma \subseteq \partial\mathbb{D}$ an open arc. If $|f(z)| \to 1$ as $z \to \gamma$, and $f(0) = 0$, then the arc length of γ is at most the arc length of $f(\gamma)$.*

Proof. Define $F : \mathbb{D} \to \mathbb{D}$ by the rule,

$$F(z) = \frac{f(\zeta z)}{f(\zeta)},$$

for some fixed $\zeta \in \gamma$. Note that F is well-defined as $|f(\zeta)| = 1$. We then see that the angular derivative at 1 yields,

$$\lim_{r\to 1} \frac{1 - F(r)}{1 - r} = F'(1) = \frac{\zeta f'(\zeta)}{f(\zeta)}.$$

Since $f'(\zeta) \neq 0$ for all $\zeta \in \gamma$, we have that $\arg f'(\zeta) = \arg f(\zeta)/\zeta$ as f is conformal. Thus,

$$\lim_{r\to 1} \frac{1 - F(r)}{1 - r} = |f'(\zeta)|.$$

Now note that by the triangle inequality and the Schwarz lemma, $|1 - F(r)| \geq 1 - |F(r)| \geq 1 - r$. Since $r \to 1$ angularly, we have that

$$|f'(\zeta)| = \lim_{r\to 1} \left|\frac{1 - F(r)}{1 - r}\right| \geq 1.$$

Since this holds for all $\zeta \in \gamma$, by integrating over γ, we have that

$$\text{length}(f(\gamma)) = \int_\gamma |f'(\zeta)|\, d\zeta \geq \int_\gamma d\zeta = \text{length}(\gamma).$$

■

Ahlfors's generalization of Schwarz-Pick

Recall that a Riemannian metric given by $ds = \rho\,|dz|$ for $\rho > 0$ is conformal with the Euclidean metric. We will only look at metrics and similar metric-concepts that are conformal with the Euclidean metric. That is, all "metrics" will be of the form $\rho\,|dz|$ for some function ρ, and may be referred to as ρ.

The (Gaussian) curvature of the Riemannian metric ρ at z is

$$K(\rho)(z) = -\frac{\Delta \log \rho(z)}{\rho(z)^2},$$

where $\Delta \coloneqq 4\frac{\partial^2}{\partial z \partial \bar{z}}$. Note that it necessary to have $\rho > 0$ and ρ twice differentiable to consider Gaussian curvature.

Example 21. The curvature of the hyperbolic metric is -1.

Example 22. The curvature of the spherical metric, $\frac{2|dz|}{1+|z|^2}$ is 1.

Example 23. Let $\Omega, \Omega' \subseteq \mathbb{C}$ be two regions, and $f : \Omega \to \Omega'$ holomorphic. If Ω' is equipped with a metric ρ, then

$$(f^*\rho)(z) = \rho(f(z))\,|f'(z)|\,|dz|$$

is the pullback of ρ under f.

Proof. This follows from the definition of pullback. ■

Curvature can tell us a lot about what a Riemannian manifold looks like. For one, positively curved spaces force geodesics to converge to one another, and negatively curved spaces forces them to spread out. Spaces of constant curvature can also be classified: if the (sectional) curvature of a complete and connected Riemannian manifold is constant, then it is isometric to M/G, where M is either Euclidean space(curvature is zero), or a sphere of radius R (curvature is $R > 0$), or hyperbolic space of radius R (curvature is $R < 0$), and G is a discrete group of isometries of M, isomorphic to $\pi_1(M)$, that acts freely and properly discontinuously on M.

Other fun facts include: If the sectional curvature of a complete and connected Riemannian manifold is bounded above by zero, then the universal covering space of M is diffeomorphic to $\mathbb{R}^n$. If the sectional curvature is bounded below by a positive constant, then M is compact, and $\pi_1(M)$ is finite.

Thus, it is an extremely nice property that curvature in our setting is preserved by pullback. Via this, we can transport hyperbolic geometry on the disk to regions not the disk!

Theorem 24. *Curvature is invariant under (holomorphic) pullback. Fix $f : \Omega \to \Omega'$ to be holomorphic, and $\Omega' \subseteq \mathbb{C}$ a region equipped with a pseudeometric $\rho(w)\,|w|$. If $\alpha \in \Omega$ is such that $f'(\alpha) \neq 0$, $\rho(f(\alpha)) > 0$, and ρ is C^2 at $f(\alpha)$, then*

$$K(f^*\rho)(\alpha) = K(\rho)(f(\alpha)).$$

Proof. Note that

$$\begin{aligned}\log(f^*\rho)(z) &= \log(\rho(f(z))\,|f'(z)|) \\ &= \log\rho(f(z)) + \frac{1}{2}\log f'(z) + \frac{1}{2}\log\overline{f'(z)}.\end{aligned}$$

It follows that,

$$\frac{\partial}{\partial z}\log f^*\rho = \frac{\rho'(f(z))\,f'(z)}{\rho(f(z))} + \frac{1}{2}\frac{f''(z)}{f'(z)} = \frac{\partial \log\rho}{\partial w}(f(z))\,f'(z) + \frac{1}{2}\frac{f''(z)}{f'(z)}.$$

By the chain rule,

$$\frac{\partial}{\partial \overline{z} \partial z} \log f^* \rho = f'(z) \frac{\partial}{\partial \overline{z}} \left(\frac{\partial \log \rho}{\partial w} (f(z)) \right) = f'(z) \left(\frac{\partial \log \rho}{\partial \overline{w} \partial w} (f(z)) \right) \frac{\partial \overline{f}}{\partial \overline{z}} = \frac{\partial \log \rho}{\partial \overline{w} \partial w} (f(z)) f'(z) \overline{f'(z)}.$$

That is, $\Delta \log f^* \rho = |f'|^2 \Delta \log \rho$. From this the claim follows shortly. ■

We now want to compare the hypebolic metric on $\mathbb{D}$, $\lambda |dz|$, with other hypebolic-esque metrics on $\mathbb{D}$, $\rho |dz|$.

Theorem 25. *If ρ satisfies $K(\rho) \leq -1$ everywhere in $\mathbb{D}$, then $\lambda(z) \geq \rho(z)$ for all $z \in \mathbb{D}$.*

Proof. First make the assumption that ρ has an extension to $\overline{\mathbb{D}}$ that is continuous and strictly positive.

From curvature, we know

$$\Delta \log \lambda = \lambda^2 \text{ and } \Delta \log \rho \geq \rho^2.$$

Thus, $\Delta(\log \lambda - \log \rho) \leq \lambda^2 - \rho^2$. Since ρ has continuous extension to the closed disk, we know that $\log \rho$ is bounded on the closed disk. It follows that $\log \lambda - \log \rho \to \infty$ as $|z| \to 1$. By compactness, $\log \lambda - \log \rho$ as a mapping into $(-\infty, \infty]$ attains a minimum $z_0 \in \overline{\mathbb{D}}$ such that $\log \lambda(z_0) - \log \rho(z_0) \in \mathbb{R}$. By the behavior at the boundary, $z_0 \in \mathbb{D}$. From calculus, it follows that $\Delta(\log \lambda - \log \rho)(z_0) \geq 0$, and therefore $\lambda(z_0)^2 \geq \rho(z_0)^2$. This implies $(\log \lambda - \log \rho)(z_0) \geq 0$. Since z_0 is a minimum, $\log \lambda - \log \rho \geq 0$ in $\mathbb{D}$ and therefore $\lambda \geq \rho$.

To remove the extra assumptions, consider $r\rho(rz)$, where $0 < r < 1$. Clearly this is the pullback of ρ under the map $z \mapsto rz$, and therefore it has the same curvature as ρ. We see that $r\rho(rz)$ extends continuously to the closed disk, as we can define the boundary values to be $r\rho(r)$. These boundary values are also strictly positive as both r and $\rho(r)$ are positive. So, $\lambda(z) \geq r\rho(rz)$ for all $z \in \mathbb{D}$. By continuity, it follows that $\lambda \geq \rho$. ■

Thus, the hyperbolic metric is the maximimal negatively curved metric on $\mathbb{D}$. Recall that when we define curvature, it is necessary that our metric is positive and twice differentiable. In some cases, this is too much to ask for, and can cause problems in applications. It is Ahlfors's observation that these regularity conditions can be avoided similarly to that of subharmonic functions to harmonic functions. Moreover, in this proof of maximality, the behavior of ρ needs to be controlled only at a single point of minimality. Perhaps then global regularity can be replaced by local regularity, leading us to the following definition:

Definition 26. A metric $\rho |dz|$ such that $\rho \geq 0$ is said to be *ultrahyperbolic* in a region Ω, if:

- ρ is upper semicontinuous.
- For all $z_0 \in \Omega$ with $\rho(z_0) > 0$, there exists a neighborhood V of z_0 and a "supporting metric" $\rho_0 \in C^2(V)$, such that $\Delta \log \rho_0 \geq \rho_0^2$ and $\rho \geq \rho_0$ in V, while $\rho(z_0) = \rho_0(z_0)$.

Note that in this definition, we must have $\rho_0 > 0$ in V. We can intuitively think of ρ_0 as a metric defined on a small ball centered at z_0 that gives V strictly negative curvature. So despite losing regularity, and not being able to define the curvature for ρ, locally we are able to give Ω some supportive type of hyperbolic structure. Moreover, ρ is given local positivity due to being bounded below by ρ_0, and at z_0, is described exactly by ρ_0.

Example 27. Any Riemannian metric that satisfies $K(\rho) \leq -1$ everywhere in a region is ultrahyperbolic, as one can take the supporting metric to be itself.

Theorem 28. *If ρ is an ultrahyperbolic metric on $\mathbb{D}$, then $\lambda \geq \rho$ everywhere on $\mathbb{D}$.*

This theorem states that λ is the maximal ultrahyperbolic metric on $\mathbb{D}$.

Proof. First make the assumption that ρ has an extension to $\overline{\mathbb{D}}$ that is upper semicontinuous.

Note that $\log \rho$ is upper semicontinuous as log is increasing and upper semicontinuous. Thus, $-\log \rho$ is lower semicontinuous, and therefore $\log \lambda - \log \rho$ is lower semicontinuous. Since ρ as upper semicontinuous extension to $\mathbb{D}$, it follows that $\log\lambda - \log\rho \to \infty$ as $|z| \to 1$. By lower semicontinuity, there exists a minimum for $\log\lambda - \log\rho$ in $\mathbb{D}$, call it z_0. Note that $\rho(z_0) \neq 0$, as otherwise z_0 is not a minimum. So $\rho(z_0) > 0$, and therefore there exists a supporting metric ρ_0 for ρ at z_0. From the definition of supporting metric,

$$\log\lambda - \log\rho_0 \geq \log\lambda - \log\rho$$

locally around z_0. So, $\log\lambda - \log\rho_0$ has local minimum at z_0 as $\rho(z_0) = \rho_0(z_0)$. This implies $\Delta(\log\lambda - \log\rho_0)(z_0) \geq 0$, and therefore $\lambda(z_0) \geq \rho_0(z_0) = \rho(z_0)$. From this, $\log\lambda(z_0) - \log\rho(z_0) \geq 0$. But since z_0 was minimum, we see that $\log\lambda(z) - \log\rho(z) \geq 0$ for all $z \in \mathbb{D}$, and therefore $\lambda \geq \rho$.

To patch the assumption of extension to the boundary, one can consider the same trick as before by looking at $r\rho(rz)$ which is still ultrahyperbolic. ■

Example 29. There is no ultrahyperbolic metric for $\mathbb{C}$.

Proof. The hyperbolic metric for a disk of radius R can be computed as,

$$\lambda_R(z) = \frac{2R}{R^2 - |z|^2}.$$

If ρ is an ultrahyperbolic metric for $\mathbb{C}$, then ρ restricts to an ultrahyperbolic metric for $\mathbb{D}_R$ and therefore $\rho \leq \lambda_R$. Taking $R \to \infty$ implies $\rho \equiv 0$. ■

Example 30. The holomorphic pullback of an ultrahyperbolic metric is ultrahyperbolic.

Proof. Clearly $\rho(f(z))|f'(z)|$ is upper semicontinuous as f' is continuous.

Fix z_0 such that $(f^*\rho)(z_0) > 0$. It follows that $\rho(f(z_0)) > 0$. Thus, there exists a supporting metric for ρ, denoted by $\rho_{f(z_0)}$ on some neighborhood V of $f(z_0)$. Define $g := f\big|_{f^{-1}(V)}$. It follows that $g^*\rho_{f(z_0)}$ satisfies negative curvature and bounds $f^*\rho$ from below while being equal at z_0. Thus, $g^*\rho_{f(z_0)}$ is a supporting metric for $f^*\rho$ at z_0. ■

We now state Ahlfors's celebrated generalization of the Schwarz-Pick theorem.

Theorem 31 (Ahlfors's lemma)**.** *Let $f \in \mathcal{O}(\mathbb{D}, \Omega)$, where Ω is equipped with an ultrahyperbolic metric ρ. Then $f^*\rho \leq \lambda$ everywhere on $\mathbb{D}$.*

Proof. Recall the invariance of ultrahyperbolicity under pullback and the maximality of the hyperbolic metric. ■

Example 32. There is no ultrahyperbolic metric on $\mathbb{C} \setminus \{0\}$.

Proof. If ρ is an ultrahyperbolic metric for $\mathbb{C} \setminus \{0\}$, then the pullback of ρ by e^z is an ultrahyperbolic metric for $\mathbb{C}$ which is a contradiction. ■

Example 33. Liouville's theorem follows immediately from Ahlfors's lemma.

Proof. If $f : \mathbb{C} \to \mathbb{C}$ is bounded, then for all $R > 0$, f maps $\mathbb{D}_R$ into some fixed $\mathbb{D}_M$. By Ahlfors's lemma, $f^*\lambda_M \leq \lambda_R$. Taking $R \to \infty$ implies $|f'| \equiv 0$. ■

Recall Schwarz-Pick: holomorphic endomorphisms of the hyperbolic disk are contractions. Ahlfors's lemma, which can be stated for Riemann surfaces states the following: holomorphic maps from the hyperbolic disk to negatively curved surfaces are contractions. As an application, we give a proof of Bloch's theorem.

Applications of Ahlfors's lemma

The first major application of Ahlfors's lemma we present is Ahlfors's proof of Bloch's theorem.

Bloch's theorem

How do holomorphic functions act when given local injectivity?

Definition 34. Fix $f \in \mathcal{O}(\mathbb{D}, \Omega)$. We say that a disk B_r with radius r in the image of f is *simple* if there exists an inverse for f on B_r. That is, there is a place where f acts biholomorphically. A disk centered at w with radius r in the image of f that is simple is denoted by $B_r(w)$. We define $B_f := \sup\{r : \exists w \in \Omega, B_r(w) \text{ is simple}\}$.

Example 35. We can translate the above definition to functions of real variables. For all $n \in \mathbb{N}$, consider $f_n : (-1,1) \to \mathbb{R}$ defined by the rule $f_n(x) = \sin(nx)/n$. Clearly each f_n is analytic and $f_n'(0) = 1$. Thus, each f_n is locally injective at 0. Moreover, $B_{f_n} = 2/n$.

The above example tells us the radii of simple disks for real-analytic functions can be arbitrarily small. What can be said about holomorphic functions from $\mathbb{D}$ to $\mathbb{C}$ that are locally injective at a point? Are there functions in this class that have arbitrarily small simple disks? What if we define $f_n(z) = e^{nz}/n$? Bloch's theorem like other theorems in complex analysis, differ from the real case.

Theorem 36. *Let $\mathscr{B} = \{f \in \mathcal{O}(\mathbb{D}) : |f'(0)| = 1\}$, and define $B = \inf\{B_f : f \in \mathscr{B}\}$. Then, $B > 0$ and in fact, $B \geq \sqrt{3}/4$.*

So, any normalized function on the disk that is locally injective at a point has a simple disk in its image with radius at least $\sqrt{3}/4$. Note that this disk need not be centered at 0 by considering example 32 except with complex entries.

Proof. Fix $f \in \mathscr{B}$, and note that $B_f > 0$ as f is locally injective at 0. Let A be a fixed constant greater than $B_f^{1/2}$. Define $R : W_f \to \mathbb{R}$ by the rule

$$R(w) = \sup\{r : B_r(w) \text{ is simple}\}.$$

We see that if $R(w) = 0$, then f is not locally injective at w. Thus, w is a point of multiplicity greater than 1. Conversely, if w has multiplicity greater than 1, then f is not locally injective, which implies $R(w) = 0$. Define $\tilde{\rho} : W_f \to \mathbb{R}$ by the rule

$$\tilde{\rho}(w) = \frac{A}{R(w)^{1/2}\left(A^2 - R(w)\right)}.$$

Clearly $B_f \geq R(w)$, and therefore $A > R(w)^{1/2}$. It follows that $\tilde{\rho}\,|dw|$ induces a metric structure on W_f. We can then equip $\mathbb{D}$ with a metric structure,

$$\rho(z) := (f^*\tilde{\rho})(z) = \frac{A\,|f'(z)|}{R(f(z))^{1/2}\left(A^2 - R(f(z))\right)}.$$

The goal is to show that ρ is ultrahyperbolic in order to apply Ahlfors's lemma.

We first consider the behavior of ρ at branch points. Suppose $w_0 = f(z_0)$ has multiplicity $n > 1$. That is, $f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0$, and $f^{(n)} \neq 0$. This implies branch points of multiplicity greater than one are isolated. It follows that there is a neighborhood of z_0, V such that $R(f(z)) = |f(z) - w_0|$ for all $z \in V$. So for $z \in V$,

$$\rho(z) = \frac{A\,|f'(z)|}{|f(z) - f(z_0)|^{1/2}\left(A^2 - |f(z) - f(z_0)|\right)}.$$

Note that $f(z) - f(z_0) = (z - z_0)^n g(z)$ where g is holomorphic and non-vanishing at z_0. Moreover, $f'(z_0) = (z - z_0)^{n-1} h(z)$ for h holomorphic and non-vanishing at z_0. It follows that,

$$\rho(z) = \frac{A\,|z - z_0|^{n-1}\,|h(z)|}{A^2\,|z - z_0|^{n/2}\,|g(z)|^{1/2} - |z - z_0|^{n/2+n}\,|g(z)|^{1/2+1}} = \frac{A\,|z - z_0|^{n/2-1}\,|h(z)|}{A^2\,|g(z)|^{1/2} - |z - z_0|^{n}\,|g(z)|^{3/2}}.$$

So for $n > 2$, ρ is continuous at z_0 (thus upper semi-continuous), and $\rho(z_0) = 0$. Therefore we need not look for a supporting metric at z_0.

For $n = 2$, we see that around a small neighborhood of z_0, that ρ is actually regular and in C^2, since g is non-vanishing at z_0 and both g and h are holomorphic. So if we can show $\Delta\rho = \rho^2$, near z_0, then ρ is ultrahyperbolic at z_0 as we can take the supporting metric to be itself. Consider $\alpha : \mathbb{D} \to \mathbb{D}$ defined by the rule,

$$\alpha(z) = A^{-1}(f(z) - f(z_0))^{1/2}.$$

Recall that $f(z) - f(z_0) = (z - z_0)^2 g(z)$, where g does not vanish at z_0. It follows that there is a small neighborhood, $G \subseteq \mathbb{D}$ around z_0 where g does not vanish. We can look at $\alpha|_G$, which is now holomorphic. It follows that $\alpha|_G{}^* \lambda = \rho$, and therefore $\Delta \log \rho = \rho^2$.

Now we must show that ρ is ultrahyperbolic on points where f is locally injective. So fix $w_0 = f(z_0)$ with $f'(z_0) \neq 0$. Consider the largest simple disk centered at w_0,

$$\Delta'(w_0) = \{w : |w - w_0| < R(w_0)\}.$$

Let $D(z_0)$ be the connected component of the preimage of $\Delta'(w_0)$ that contains z_0. We claim that there exists $a \in \partial D(z_0)$ such that $f'(a) = 0$, or $|a| = 1$ (see the appendix on Bloch's theorem for proof). Note that in the first case, the boundary of $\Delta'(w_0)$ contains $b = f(a)$. In the second case, we may assume that f extends continuously to $\partial\mathbb{D}$ by a standard rescaling argument. In either case, $b = f(a)$ is on the boundary of $\Delta'(w_0)$.

Fix $z_1 \in D(z_0)$, and let $w_1 = f(z_1)$. We claim that $R(w_1) \leq |w_1 - b|$. This is intuitively a contradiction as b is either a point that fails local injectivity, or a boundary point, and therefore cannot be contained in a simple disk. For a formal argument, see the appendix on Bloch's theorem.

We claim that a supporting metric for ρ at z_0 is defined to be

$$\rho_0(z) := \frac{A|f'(z)|}{|f(z) - b|^{1/2}(A^2 - |f(z) - b|)},$$

for z close to z_0. Clearly ρ_0 satisfies positivity, and is regular on a small neighborhood of z_0. Recalling the argument for showing ultrahyperbolicity of ρ at points with multiplicity 2, we see that ρ_0 has curvature -1. Since $R(f(z_0) =)|f(z_0) - b|$, we have that $\rho_0(z_0) = \rho(z_0)$. Monotonicity of the metrics comes down to showing that for z close to z_0,

$$R(f(z))^{1/2}\left(A^2 - R(f(z))\right) \leq |f(z) - b|^{1/2}\left(A^2 - |f(z) - b|\right).$$

So it suffices to look at when the function: $t^{1/2}\left(A^2 - t\right)$ is increasing for $0 \leq t \leq R(w_0)$, as we have equality at $t = R(w_0)$, and $R(f(z)) \leq R(w_0)$ for z close to z_0. We see that $t^{1/2}\left(A^2 - t\right)$ is increasing for $0 \leq t \leq A^2/3$. Thus, if we take $A^2/3 > B_f \geq R(w)$, it follows that ρ_0 is a supporting metric for ρ at z_0, and therefore ρ is an ultrahyperbolic metric on $\mathbb{D}$. By Ahlfors's lemma, for all $z \in \mathbb{D}$,

$$\rho(z) \leq \lambda(z).$$

By evaluating at $z = 0$ and our choice of A before,

$$A \leq 2R(f(z))^{1/2}\left(A^2 - R(f(z))\right) \leq 2B_f^{1/2}\left(A^2 - B_f\right).$$

Letting A tend to $(3B_f)^{1/2}$, we see that $B_f \geq \sqrt{3}/4$. Thus, $B \geq \sqrt{3}/4$. ■

Bounds for Poincaré metrics

It is always useful to obtain explicit bounds on functions that cannot be expressed explicitly. In previous examples we showed that there are no ultrahyperbolic metrics on $\mathbb{C}$ or $\mathbb{C} \setminus \{0\}$. These are the only two examples of such domains.

Theorem 37. *In a plane region Ω whose complement contains at least two points, there exists a unique maximal ultrahyperbolic metric λ_Ω with constant curvature -1.*

Maximality is given in the sense that for any other ultrahyperbolic metric ρ on Ω, we have $\rho \leq \lambda_\Omega$. We do not prove this theorem. Existence follows from the uniformization theorem, and uniqueness is immediate. Moreover, we define this maximal ultrahyperbolic metric on Ω to be the *Poincaré metric* for Ω.

Example 38. Poincaré metrics are preserved by biholomorphisms. That is, if $f : \Omega \to \Omega'$ is a biholomorphism, and $\lambda_{\Omega'}$ is the Poincaré metric for Ω', then $f^*(\lambda_{\Omega'})$ is the Poincaré metric for Ω.

Proof. Fix ρ ultrahyperbolic on Ω. It follows that $f^{-1*}\rho \leq \lambda_{\Omega'}$. So, $f^*\left(f^{-1*}\rho\right) \leq f^*\lambda_\Omega$. Note however the lefthand side is ρ. Thus, $f^*\lambda_\Omega$ is maximal and has constant curvature -1. ∎

It is easy to obtain upper bounds for Poincaré metrics.

Theorem 39. *If $\Omega \subseteq \Omega'$, then $\lambda_{\Omega'} \leq \Lambda_\Omega$.*

Proof. The restriction of an ultrahyperbolic metric is ultrahyperbolic. ∎

Theorem 40. *Let $\delta(z)$ be the distance from $z \in \Omega$ to $b\Omega$. It follows that $\lambda_\Omega(z) \leq 2/\delta(z)$.*

Proof. Note that the disk centered around z of radius $\delta(z)$ is contained in Ω. By using the hyperbolic metric on a disk, and the previous theorem, it follows that $\lambda_\Omega(z) \leq 2/\delta(z)$. ∎

Note that the above estimate is the best upper bound for general Ω as we have equality in the case of a disk.

It is much more difficult to obtain lower bounds.

Since every domain that admits an ultrahyperbolic metric is contained in some $\Omega_{a,b} = \mathbb{C} \setminus \{a, b\}$, it suffices to find a lower bound for the Poincaré metric $\lambda_{a,b}$ on $\Omega_{a,b}$. Moreover, since

$$\lambda_{a,b}(z) = |b-a|^{-1}\lambda_{0,1}\left(\frac{z-a}{b-a}\right),$$

it suffices to find a lower bound for $\lambda_{0,1}$. Optimally, we want this lower bound to be expressed in elementary terms. To do this, we break up $\Omega_{0,1}$ into simpler pieces. Let

$$\Omega_1 = \{|z| \leq 1 : |z| \leq |z-1|\},$$

and Ω_2 the reflection of Ω_1 over $x = 1/2$, and Ω_3 the closure of the complement of $\Omega_1 \cup \Omega_2$. This decomposition is motivated by the fact that $\Omega_{0,1}$ has automorphisms $1-z$ and $1/z$. Thus, Ω_1, Ω_2, and Ω_3 are fundamental domains and therefore we only need to consider bounds for $\lambda_{0,1}$ in either Ω_1, Ω_2, or Ω_3.

We now look for a stronger upper bound. Note that the Poincaré metric on the punctured disk $\mathbb{D} \setminus \{0\}$ is such an upper bound. Since the left half-plane with the map $z = e^w$ is a holomorphic universal cover for the punctured disk, we can compute the Poincaré metric on $\mathbb{D} \setminus \{0\}$ from the Poincaré metric on the left half-plane (see Beardon and Minda's notes). Thus,

$$\frac{|dw|}{|\operatorname{Re} w|} = \frac{|d\log z|}{|\operatorname{Re}\log z|} = \frac{|dz|}{|z|\,|\log|z||} = \frac{|dz|}{|z|\log\frac{1}{|z|}}$$

is the Poincaré metric on $\mathbb{D} \setminus \{0\}$. It follows that,

$$\lambda_{0,1}(z) \leq \frac{1}{|z|\log\frac{1}{|z|}}$$

for all $z \in \mathbb{D} \setminus \{0\}$.

To state an elementary lower bound on $\lambda_{0,1}$ we need to define the following function. Let $\zeta : \mathbb{C}\setminus[1,\infty] \to \mathbb{D}$ be defined by the rule,

$$\zeta(z) = \frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}.$$

Geometrically, this function conformally maps $\mathbb{C} \setminus [1, \infty]$ first to the right half-plane, and then into the unit disk. Moreover, the origins are mapped to each other, and symmetry with respect to the x-axis is preserved. That is, $\zeta(\overline{z}) = \overline{\zeta(z)}$.

Theorem 41. *For $z \in \Omega_1$,*

$$\lambda_{0,1}(z) \geq \left|\frac{\zeta'(z)}{\zeta(z)}\right| (4 - \log|\zeta(z)|)^{-1}.$$

Moreover, as $z \to 0$,

$$\log \lambda_{0,1}(z) = -\log|z| - \log\log\frac{1}{|z|} + O(1).$$

Proof. Clearly $\mathbb{D}\setminus\{0\}$ is biholomorphic to the punctured disk of radius e^4. Thus, we can equip this punctured disk with the metric $(|z|(4-\log|z|))^{-1}$. Since ζ maps into the punctured disk of radius e^4 via the standard inclusion, by pulling back we see that

$$\rho_0(z) = \left|\frac{\zeta'(z)}{\zeta(z)}\right| (4 - \log|\zeta(z)|)^{-1}$$

is an ultrahyperbolic metric on $\mathbb{C}\setminus[1,\infty]$ with curvature -1. In particular, we regard it is an ultrahyperbolic metric on Ω_1. To obtain a lower bound for $\lambda_{0,1}$ we must show that ρ_0 can be extended to an ultrahyperbolic metric on $\Omega_{0,1}$. This can be done by extending to Ω_2 and Ω_3 via symmetry. The candidate function $\rho : \Omega_{0,1} \to \mathbb{R}_{\geq 0}$ is given by the rule,

$$\rho(z) = \begin{cases} \rho_0(z) & z \in \Omega_1 \\ \rho_0(1-z) & z \in \Omega_2 \\ |z|^{-2}\rho_0\left(\frac{1}{z}\right) & z \in \Omega_3 \end{cases}.$$

Clearly ρ has constant curvature -1 as for $z \in \Omega_1$ it is defined by ρ_0, and for the other parts of the domain, the rest of the metric is formed by the pullback.

We now show that ρ is continuous/well-defined. This follows essentially from the fact that $1/z$ and $1-z$ map the boundaries of Ω_1, Ω_2, and Ω_3 to one another. To elaborate, we consider $\Omega_1 \cap \Omega_2$, $\Omega_1 \cap \Omega_3$, and $\Omega_2 \cap \Omega_3$. If $z \in \Omega_1 \cap \Omega_2$ we need to show $\rho(z)$ makes sense. That is, $\rho_0(z) = \rho_0(1-z)$. Note that $z \in \Omega_1 \cap \Omega_2$ if and only if $z = 1/2 + ib$ for some real b. We see that $\rho_0(z) = \rho_0(1-z)$ if and only if

$$\rho_0\left(1 - \left(\frac{1}{2} + ib\right)\right) = \rho_0\left(\frac{1}{2} - ib\right) = \rho_0\left(\frac{1}{2} + ib\right).$$

This latter statement is true since ρ_0 is defined in terms of ζ, and ζ has symmetry about the x-axis and therefore so does its derivative.

Now consider $z \in \Omega_1 \cap \Omega_3$. This implies $|z| = 1$. In this case we see that $\rho_0(z) = |z|^{-2}\rho_0(1/z)$ as $1/z = \bar{z}$, and ζ is symmetric. Finally, consider $z \in \Omega_2 \cap \Omega_3$. One can show that in this case that

$$\operatorname{Re}\zeta(1/z) = -\operatorname{Re}\zeta(1-z) \text{ and } \operatorname{Im}\zeta(1/z) = \operatorname{Im}\zeta(1-z).$$

This implies $|\zeta(1/z)| = |\zeta(1-z)|$. Moreover, one can show that for $z \in \Omega_2 \cap \Omega_3$,

$$\operatorname{Re}\frac{z}{|z|^2\sqrt{1-1/z}} = \operatorname{Im}\frac{1}{(1-z)\sqrt{z}} \text{ and } \operatorname{Im}\frac{z}{|z|^2\sqrt{1-1/z}} = -\operatorname{Re}\frac{1}{(1-z)\sqrt{z}}.$$

These computations imply the last case, and therefore continuity of the whole metric.

To show ultrahyperbolicity, we now need to show there exists a supporting metric. Note that there is already a supporting metric for ρ on the interiors of Ω_1, Ω_2, and Ω_3 as ρ was given by the pullback of a Poincaré metric, and it was extended via further pullbacks. Thus, we only have to show there is a supporting metric on the lines separating Ω_1, Ω_2, and Ω_3. We claim that if we can the existence of such a support r on $\Omega_1 \cap \Omega_2$, then we are done. This is because if $z \in \Omega_2 \cap \Omega_3$, then $|z|^{-2} r(1/z)$ is a supporting metric induced by the one on $\Omega_1 \cap \Omega_2$. Similarly, if $z \in \Omega_1 \cap \Omega_3$, then $r(1-z)$ is a supporting metric induced by the one on $\Omega_2 \cap \Omega_3$.

We claim that the original ρ_0 is a supporting metric for ρ on $\Omega_1 \cap \Omega_2$. One needs only to show $\rho_0(z) \leq \rho(z)$ locally, and have equality on $\Omega_1 \cap \Omega_2$. Clearly we have such equality. To show the inequality, we claim that it suffices to know $\partial\rho_0/\partial x < 0$ on $\Omega_1 \cap \Omega_2$. If $\partial\rho_0/\partial x < 0$, then for any $z \in \Omega_1 \cap \Omega_2$, there is a small horizontal line centered at z so that ρ_0 is decreasing. We can then consider the union of these strips around some fixed z to make an open neighborhood U containing z. Clearly $\rho_0(z) = \rho(z)$. Moreover, for any point in $\Omega_1 \cap U$, $\rho_0 = \rho$. Similarly, for any point w in $\Omega_2 \cap U$, since ρ_0 was decreasing on these strips, we have that $\rho_0(w) \leq \rho_0(1-w) = \rho(w)$ as $1-w$ is mapped back into Ω_1. This implies $\rho_0(z) \leq \rho(z)$ for all $z \in U$, and therefore ρ_0 is a suitable supporting metric on $\Omega_1 \cap \Omega_2$.

We now show that $\partial\rho_0/\partial x < 0$. This is equivalent to showing $\partial \log\rho_0/\partial x < 0$. Since $\log\rho_0(z) = \log|\zeta'(z)/\zeta(z)| - \log(4 - \log|\zeta(z)|)$, we have that

$$\frac{\partial \log \rho_0}{\partial x} = \operatorname{Re}\left(\frac{d}{dz}\log\left|\frac{\zeta'(z)}{\zeta(z)}\right|\right) + \operatorname{Re}\left(\left|\frac{\zeta'(z)}{\zeta(z)}\right|(4-\log|\zeta(z)|)^{-1}\right).$$

Since

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{1}{z\sqrt{1-z}} \text{ and } \frac{d}{dz}\log\frac{\zeta'(z)}{\zeta(z)} = \frac{3z-2}{2z(1-z)},$$

and $1 - z = \overline{z}$ on $\Omega_1 \cap \Omega_2$, we have that

$$\begin{aligned}
\frac{\partial \log\rho_0}{\partial x}(z) &= \operatorname{Re}\left(\frac{3z-2}{2z\overline{z}}\right) + \operatorname{Re}\frac{\sqrt{\overline{z}}}{z\overline{z}}(4-\log|\zeta(z)|)^{-1} \\
&= -\frac{1}{4|z|^2} + \operatorname{Re}\frac{\sqrt{z}}{|z|^2}(4-\log|\zeta(z)|)^{-1} \\
&\leq -\frac{1}{4|z|^2} + \frac{1}{|z|^2}(4-\log|\zeta(z)|)^{-1} \\
&< 0
\end{aligned}$$

as $\log|\zeta(z)|$ is negative, and $\operatorname{Re}\sqrt{z} \leq 1$. Thus, ρ_0 is a supporting metric on $\Omega_{0,1}$ and therefore ρ is ultrahyperbolic. Thus, ρ serves as a lower bound for $\lambda_{0,1}$.

The second inequality is a straightforward computation. We know

$$\left|\frac{\zeta'(z)}{\zeta(z)}\right|(4-\log|\zeta(z)|)^{-1} \leq \lambda_{0,1}(z) \leq \frac{1}{|z|\log\frac{1}{|z|}}$$

for all $z \in \Omega_1$. It follows that,

$$\log\left(\left|\frac{\zeta'(z)}{\zeta(z)}\right|(4-\log|\zeta(z)|)^{-1}\right) \leq \log\lambda_{0,1}(z) \leq \log\left(\frac{1}{|z|\log\frac{1}{|z|}}\right) = -\log|z| - \log\log\frac{1}{|z|}.$$

This implies

$$\log\left(\left|\frac{\zeta'(z)}{\zeta(z)}\right||z|(4-\log|\zeta(z)|)^{-1}\log\frac{1}{|z|}\right) \leq \log\lambda_{0,1}(z) + \log|z| + \log\log\frac{1}{|z|} \leq 0.$$

Now as $z \to 0$ we see that,

$$\left|\frac{\zeta'(z)}{\zeta(z)}\right||z| \to 1 \text{ and } -\frac{\log|z|}{4-\log|\zeta(z)|} \to 1$$

by the definition of the derivative and L'Hôpital's rule. Thus, on the lefthand side the inequality tends to 0, and therefore $\log\lambda_{0,1}(z) = -\log|z| - \log\log\frac{1}{|z|} + O(1)$ as $z \to 0$. ■

Note that the reason why e^4 was chosen in the beginning of the proof is made clear when justifying $\partial\log\rho_0/\partial x < 0$ in the very last step.

The Picard theorems

As an application of the bounds obtained on Poincaré metrics, and Ahlfors's lemma, we prove an explicit quantitative version of Picard's little theorem.

Theorem 42 (Schottky)**.** *Suppose that $f \in \mathcal{O}(\mathbb{D})$ omits the values 0 and 1. Then*

$$\log|f(z)| \leq (7+\max(0,\log|f(0)|))\,\frac{1+|z|}{1-|z|}.$$

Proof. We know that f and $1/f$ satisfy the same assumptions, and therefore we can obtain an upper bound or a lower bound for $|f|$ to obtain our result. In our case, we look for a lower bound. By Ahlfors's lemma we know that

$$\lambda_{0,1}(f(z))\,|f'(z)| \leq \frac{2}{1-|z|^2}.$$

By integrating along the straight line from 0 to some $z \in \mathbb{D}$,

$$\int_{f(0)}^{f(z)} \lambda_{0,1}(w)\ |dw| = \int_0^z \lambda_{0,1}(f(t))\,|f'(t)|\ |dt| \leq \int_0^z \frac{2}{1-|t|^2}\ |dt| = \log\frac{1+|z|}{1-|z|}.$$

There are two cases to consider: whether or not the image of the straight line lives in Ω_1 (the same one defined in the last section).First assume that the image of the straight line lives in Ω_1. Then by the estimate obtained in the last section, we have that

$$\int_{f(0)}^{f(z)} (4-\log|\zeta(w)|)^{-1}\,|d\log\zeta(w)| = \int_{f(0)}^{f(z)} \left|\frac{\zeta'(w)}{\zeta(w)}\right| (4-\log|\zeta(w)|)^{-1}\ |dw| \leq \log\frac{1+|z|}{1-|z|}.$$

Now since $d\log|\zeta| \geq 0$, we have that $-d\log|\zeta| \leq |d\log\zeta|$. Thus,

$$-\int_{f(0)}^{f(z)} (4-\log|\zeta(w)|)^{-1}\ d\,|\log\zeta(w)| \leq \log\frac{1+|z|}{1-|z|},$$

and therefore

$$\frac{4-\log|\zeta(f(z))|}{4-\log|\zeta(f(0))|} \leq \frac{1+|z|}{1-|z|}.$$

Recalling the explicit form of ζ, we see that

$$|\zeta(w)| = \frac{|w|}{\left|\sqrt{1-w}+1\right|^2}.$$

Now recall that $f(0), f(z) \in \mathbb{D}$. Let $w \in \mathbb{D}$ be arbitrary. Since $\operatorname{Re}\sqrt{1-w} > 0$ we have that $|\zeta(w)| \leq |w|$. Moreover, since $1-w$ is in the open disk of radius 1 centered at 1, we have that $\left|\sqrt{1-w}\right|$ is bounded above by $\sqrt{2}$. Thus, $\left|\sqrt{1-w}+1\right| \leq 1+\sqrt{2}$, which implies $\left(1+\sqrt{2}\right)^{-2}|w| \leq |\zeta(w)|$. It follows that,

$$4-\log|f(z)| \leq 4-\log|\zeta(f(z))| \leq (4-\log|\zeta(f(0))|)\,\frac{1+|z|}{1-|z|} \leq \left(4-\left(-2\log\left(1+\sqrt{2}\right)+\log|f(0)|\right)\right)\frac{1+|z|}{1-|z|}.$$

Then since $\log\left(1+\sqrt{2}\right) < 1$, we have that

$$\log|1/f(z)| \leq (6+\log|1/f(0)|)\,\frac{1+|z|}{1-|z|}.$$

Now consider the case in which the image of the line is not contained in Ω_1. We then have $f(z) \in \Omega_1$ or $f(z) \notin \Omega_1$. If $f(z) \in \Omega_1$, then there exists a point w_0 in the image of the line that intersects the boundary of Ω_1. Moreover, we can pick w_0 to be the last point that intersects the boundary. In this case, we would have that

$$\int_{w_0}^{f(z)} (4-\log|\zeta(w)|)^{-1}\,|d\log\zeta(w)| \leq \int_{f(0)}^{f(z)} (4-\log|\zeta(w)|)^{-1}\,|d\log\zeta(w)|.$$

Then analogously we would obtain,

$$-\log|f(z)| \leq (6-\log|w_0|)\,\frac{1+|z|}{1-|z|}.$$

Note however in this case we would have

$$\log |1/f(z)| \leq (6 + \log 2) \frac{1 + |z|}{1 - |z|}$$

as $|w_0| \geq 1/2$.

Now consider the case where $f(z) \notin \Omega_1$. It follows that $|f(z)| \geq 1/2$, and therefore $\log 2 > \log |1/f(z)|$. However, $(1 + |z|) / (1 - |z|) \geq 1$, and therefore the above inequality is obtained trivially as $\log 2 < 1$. Combining the two cases we see that

$$\log |1/f(z)| \leq (6 + \log 2 + \max(0, \log |1/f(z)|)) \frac{1 + |z|}{1 - |z|}.$$

We can then replace f with $1/f$ and we then see that this inequality implies the one in the stated theorem. ∎

We now apply Schottky's theorem to obtain information about the behavior of meromorphic functions on the plane. We know from Liouville's theorem that bounded entire functions are constant. Moreover, one can show that images of non-constant entire functions are dense in $\mathbb{C}$. From the Casorati-Weierstrass theorem, we know the behavior of holomorphic functions about essential singularities. That is, for any essential singularity, the image of any punctured neighborhood about that singularity is dense in $\mathbb{C}$.

Theorem 43 (The little Picard theorem)**.** *If f is meromorphic in the plane and omits three values, then f is constant.*

Proof. Let a, b, c be the three values f omits. It follows that

$$F(z) = \frac{c - b}{c - a} \frac{f(z) - a}{f(z) - b}$$

is entire, and omits 0 and 1. Now fix $R > 0$, and define g from the unit disk to the disk of radius R by the rule $g(z) = Rz$. By Schottky's theorem,

$$\log |F(g(z))| \leq (7 + \max(0, \log |F(0)|)) \frac{1 + |z|}{1 - |z|}.$$

It follows that

$$\log \left| F\left(R \frac{e^{i\theta}}{2}\right) \right| \leq 3 (7 + \max(0, \log |F(0)|)).$$

Since $F\left(Re^{i\theta}/2\right)$ is bounded above by a finite constant independent of R and θ, we have that F is constant and therefore so is f. ∎

Note that from the above proof the little Picard theorem for entire functions follows immediately as we only apply Schottky's theorem to an entire function that omits two values.

Theorem 44 (The big Picard theorem)**.** *If f is meromorphic and omits three values in a punctured disk $0 < |z| < \delta$, then it has a meromorphic extension to the full disk.*

Proof. Without loss of generality assume that $\delta = 1$ and that f omits $0, 1, \infty$. Since we want to show f has a meromorphic extension to the whole disk, it is sufficient to show good behavior of f in a small neighborhood of 0. Fix $|z| < 1/4$ and $|z_0| = 1/2$ such that $\arg z = \arg z_0$. Note that if $f(z) \notin \Omega_1$, we have that $1/2 \leq |f(z)|$. This implies f is well-behaved on these values of z. So assume $f(z) \in \Omega_1$. Since f is defined in the punctured disk and maps into $\Omega_{0,1}$, we have that

$$\lambda_{0,1}(f(z)) |f'(z)| \leq \frac{1}{|z| \log \frac{1}{|z|}}.$$

By integrating about the straight line γ from z_0 to z we have that,

$$\int_{z_0}^{z} \lambda_{0,1}(f(w)) |f'(w)| \, |dw| \leq \int_{z_0}^{z} \frac{|dw|}{|w| \log \frac{1}{|w|}} = \int_0^1 \frac{|\gamma'(t)|}{|\gamma(t)| \log \frac{1}{|\gamma(t)|}} \, dt.$$

Since

$$\frac{d}{dt}\log\log\frac{1}{|\gamma(t)|}=\frac{|\gamma'(t)|}{|\gamma(t)|\log\frac{1}{|\gamma(t)|}}$$

we have that

$$\int_{f(z_0)}^{f(z)}\lambda_{0,1}(t)\,|dt|=\int_{z_0}^{z}\lambda_{0,1}(f(w))\,|f'(w)|\,|dw|\leq\log\log\frac{1}{|z|}-\log\log\frac{1}{1/2}\leq\log\log\frac{1}{|z|}.$$

By the same argument as in the proof of Schottky's theorem, we have that

$$\log(4-\log|f(z)|)\leq\log\log\frac{1}{|z|}+\log\left(4+2\log\left(1+\sqrt{2}\right)-\log|w_0|\right)$$

where $w_0\in\Omega_1$. Since $|w_0|\leq 1$, we have that

$$-\log|f(z)|\leq C\log\frac{1}{|z|}$$

for some integer $C>0$ independent of z. This implies $1\leq|f(z)|/|z|^C$ for z such that $f(z)\in\Omega_1$. But recall that $1/2\leq|f(z)|$ for z such that $f(z)\notin\Omega_1$. In particular this implies $|f(z)|/|z|^C$ is bounded below by some positive number independent of z. By the Casorati-Weierstrass theorem we have that the singularity at 0 for $f(z)/z^C$ is not essential, and therefore f has meromorphic extension to $\mathbb{D}$. ■

Appendix

Nevanlinna-Pick interpolation

The following theorem proven by Georg Pick and Rolf Nevanlinna independently in 1916 and 1919 respectively is an n-point generalization of the Schwarz-Pick theorem. It answers the following question: given *initial data* $z_1, \ldots, z_n \in \mathbb{D}$ and *target data* $w_1, \ldots, w_n \in \mathbb{D}$, when does there exists $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ so that $f(z_i) = w_i$ for all $1 \leq i \leq n$?

Theorem 45 (Nevanlinna-Pick Interpolation)**.** *Let $z_1, z_2, \ldots, z_n$ be our initial data in $\mathbb{D}$, and $w_1, w_2, \ldots, w_n$ our target data in $\mathbb{D}$. Then there exists a holomorphic $f : \mathbb{D} \to \overline{\mathbb{D}}$ that interpolates our data if and only if the matrix*

$$\left(\frac{1 - w_j \overline{w}_k}{1 - z_j \overline{z}_k}\right)_{i,j=1}^{n}$$

is positive semi-definite.

Note that this is a clear generalization of the Schwarz-Pick theorem. For $n = 2$, if $z_1, z_2 \in \mathbb{D}$, $f \in \mathcal{O}(\mathbb{D}, \mathbb{D})$, and $w_1 = f(z_1)$ and $w_2 = f(z_2)$, then

$$\begin{aligned}
\delta(w_1, w_2) &\leq \delta(z_1, z_2) \\
1 - \delta(z_1, z_2) &\leq 1 - \delta(w_1, w_2) \\
\frac{1 - |z_1|^2}{1 - z_1 \overline{z}_2} \cdot \frac{1 - |z_2|^2}{1 - \overline{z}_1 z_2} &\leq \frac{1 - |w_1|^2}{1 - w_1 \overline{w}_2} \cdot \frac{1 - |w_2|^2}{1 - \overline{w}_1 w_2} \\
\frac{1 - w_1 \overline{w}_2}{1 - z_1 \overline{z}_2} \cdot \frac{1 - \overline{w}_1 w_2}{1 - \overline{z}_1 z_2} &\leq \frac{1 - |w_1|^2}{1 - |z_1|^2} \cdot \frac{1 - |w_2|^2}{1 - |z_2|^2} \\
0 &\leq \begin{vmatrix} \frac{1-|w_1|^2}{1-|z_1|^2} & \frac{1-|w_2|^2}{1-|z_2|^2} \\ \frac{1-|w_1|^2}{1-|z_1|^2} & \frac{1-|w_2|^2}{1-|z_2|^2} \end{vmatrix}.
\end{aligned}$$

Note that the above is a string of if and only if statements. To complete the proof for the $n = 2$ case of Nevanlinna-Pick interpolation, we would have to find an explicit f that interpolates the data.

We only prove the forward direction for the higher dimensional case. To do this we will need an integral representation which will be proven at the end of the section.

Proof. Set $F(z) = \frac{1+f(z)}{1-f(z)}$. Note that F has positive real part. Set $F = U + iV$. We have that,

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} U\left(e^{i\theta}\right) \, d\theta + iV(0).$$

It follows that

$$\begin{aligned}
F(z_h) + \overline{F(z_k)} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} U\left(e^{i\theta}\right) + \frac{e^{-i\theta} + \overline{z}_k}{e^{-i\theta} - \overline{z}_k} U\left(e^{i\theta}\right) \, d\theta \\
&= \frac{1}{\pi} \int_0^{2\pi} \frac{1 - z_h \overline{z}_k}{\left(e^{i\theta} - z_h\right)\left(e^{-i\theta} - \overline{z}_k\right)} U\left(e^{i\theta}\right) \, d\theta.
\end{aligned}$$

Now note that

$$\left|\sum_{k=1}^{n} \alpha_k\right|^2 = \sum_{i,j=1}^{n} \alpha_i \overline{\alpha}_j.$$

It follows that,

$$\begin{aligned}\sum_{h,k=1}^{n} \frac{F(z_h)+\overline{F(z_k)}}{1-z_h\overline{z}_k} t_h\bar{t}_k &= \sum_{h,k=1}^{n} \frac{1}{\pi}\int_0^{2\pi} \frac{t_h\bar{t}_k}{\left(e^{i\theta}-z_h\right)\left(e^{-i\theta}-\overline{z}_k\right)} U\left(e^{i\theta}\right)\, d\theta \\ &= \frac{1}{\pi}\int_0^{2\pi} U\left(e^{i\theta}\right) \sum_{h,k=1}^{n} \frac{t_h\bar{t}_k}{\left(e^{i\theta}-z_h\right)\left(e^{-i\theta}-\overline{z}_k\right)} U\left(e^{i\theta}\right)\, d\theta \\ &= \frac{1}{\pi}\int_0^{2\pi} \left|\sum_{j=1}^{n} \frac{t_j}{e^{i\theta}-z_j}\right|^2 U\left(e^{i\theta}\right)\, d\theta \geq 0\end{aligned}$$

as $U \geq 0$. Moreover, since $F = \frac{1+f}{1-f}$, we see that $F(z_h)+\overline{F(z_k)} = 2\left(1-w_h\overline{w}_k\right)/\left(\left(1-w_h\right)\left(1-\overline{w}_k\right)\right)$. It follows that our Hermitian form is semi-positive definite. ∎

We now give the proofs of the integral representation for holomorphic functions.

Lemma 46. Suppose $f \in \mathcal{O}(\mathbb{D})$. Then for all $z \in \mathbb{D}$ we have that

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\overline{f(w)}}{w-z}\, dw = \overline{f(0)}.$$

Proof. Since f is holomorphic, we know f has power series representation

$$f(w) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} w^k.$$

Parametrizing $\partial\mathbb{D}$ by $w = e^{i\theta}$ for $0 \leq \theta \leq 2\pi$, we see that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\overline{f(w)}}{w-z}\, dw &= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\sum_{k=0}^{\infty}\overline{\frac{f^{(k)}(0)}{k!}}\,\overline{w}^k}{w-z}\, dw \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\sum_{k=0}^{\infty}\overline{\frac{f^{(k)}(0)}{k!}}e^{-i\theta k}}{e^{i\theta}-z} ie^{i\theta}\, d\theta. \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\sum_{k=0}^{\infty}\overline{\frac{f^{(k)}(0)}{k!}}e^{-i\theta k}}{1-ze^{-i\theta}}\, d\theta.\end{aligned}$$

Since $z \in \mathbb{D}$ we can use the geometric series. It follows that,

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\overline{f(w)}}{w-z}\, dw = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{k=0}^{\infty} \frac{\overline{f^{(k)}(0)}}{k!} e^{-i\theta k}\right)\left(\sum_{n=0}^{\infty} e^{-in\theta} z^n\right) d\theta.$$

Now recall that if $m \in \mathbb{Z}$, we have

$$\int_0^{2\pi} e^{im\theta}\, d\theta = \begin{cases} 2\pi & m = 0 \\ 0 & m \neq 0 \end{cases}.$$

So it follows that

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{\overline{f(w)}}{w-z}\, dw = \frac{1}{2\pi}\int_0^{2\pi} \overline{f(0)}\, d\theta = \overline{f(0)}.$$

∎

Theorem 47. *Suppose $f = u + iv \in \mathcal{O}(\mathbb{D})$. Then,*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} u\left(e^{i\theta}\right)\, d\theta + iv(0).$$

Proof. By the above lemma and some algebra, we know

$$\begin{aligned}
f(z) + \overline{f(0)} &= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{f(w) + \overline{f(w)}}{w - z}\, dw \\
&= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{2u(w)}{w - z}\, dw \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{2u\left(e^{i\theta}\right)}{e^{i\theta} - z} e^{i\theta}\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{i\theta}\right) \frac{e^{i\theta} + z + e^{i\theta} - z}{e^{i\theta} - z}\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} u\left(e^{i\theta}\right)\, d\theta + \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{i\theta}\right)\, d\theta
\end{aligned}$$

By the mean value property for harmonic functions, we know

$$f(z) + \overline{f(0)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} u\left(e^{i\theta}\right)\, d\theta + u(0).$$

Then since $\overline{f(0)} = u(0) - iv(0)$, we have our claim. ■

Two lemmas for Bloch's theorem

Lemma 48. Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic. Furthermore, assume there exists an open subset $U \subseteq \mathbb{D}$ such that $\overline{U} \subseteq \mathbb{D}$ and f restricted to U is a biholomorphism with image a disk, and $f'(z) \neq 0$ for all $z \in \partial U$. It follows that there exists an open subset $W \subseteq \mathbb{D}$ such that f remains a biholomorphism, and $\overline{U} \subseteq W \subseteq \overline{W} \subseteq \mathbb{D}$.

Proof. Since U is bounded, by this stackexchange answer, f is injective on $\overline{U}$.

Fix $z \in \partial U$. We claim that there exists $\varepsilon > 0$ such that f is injective on $U \cup B_\varepsilon(z)$. Suppose not, then for all sufficiently large $n \in \mathbb{N}$, there exists $a_n \in U \setminus B_{1/n}(z)$ and $b_n \in B_{1/n}(z) \setminus U$ such that $b_n \to z$, and $f(a_n) = f(b_n)$. By continuity, $f(b_n) \to f(z)$. Moreover, since $\overline{U}$ is compact, there exists a convergent subsequence a_{n_k}. It follows that,

$$f(a_{n_k}) = f(b_{n_k}) \to f(z).$$

By injectivity on $\overline{U}$, $a_{n_k} \to z$, and therefore any limit point of $\{a_n\}$ is z, implying $a_n \to z$. Now consider any neighborhood around z that admits an injective restriction for f. It follows that for large enough n, both a_n and b_n live in the neighborhood. This contradicts the local injectivity at z, and therefore there exists $\varepsilon > 0$ such that f is injective on $U \cup B_\varepsilon(z)$.

Now suppose for contradiction that there exists no open $W \subseteq \mathbb{D}$ so that f remains a biholomorphism, and $\overline{U} \subseteq W \subseteq \overline{W} \subseteq \mathbb{D}$. Define $U_n = \{z \in \mathbb{D} : d(z, U) < 1/n\}$. It follows that for sufficiently large n, for any $z_n \in U_n \setminus \overline{U}$, there exists $w_n \in U_n$ so that $w_n \neq z_n$, and $f(z_n) = f(w_n)$.

For sufficiently large n, pick $z_n \in U_n \setminus \overline{U}$ so that $z_n \to z$ is a convergent sequence. Note that $z \in \partial U$. Let w_n be the corresponding sequence so that $w_n \neq z_n$, and $f(z_n) = f(w_n)$. Now let w_{n_k} be a subsequence that converges to $w \in \overline{U}$. Note that this implies $f(z) = f(w)$. Since f is a homeomorphism on $\overline{U}$, it follows that $z = w$. However, this contradicts the injectivity of $U \cup B_\varepsilon(z)$ for some sufficiently small ε. ■

This lemma can then be applied to the situation we had in the proof for Bloch's theorem. We extend $D(z_0)$ to an open set W such that f is injective on W, and $\overline{D(z_0)} \subseteq W \subseteq \overline{W} \subseteq \mathbb{D}$. This contradicts maximality of $\Delta'(w_0)$, as the image of W, which is open, contains $\overline{\Delta'(w_0)}$.

Lemma 49. If $z_1 \in D(z_0)$, and $w_1 = f(z_1)$, then $R(w_1) \leq |w_1 - b|$.

Proof. Consider $\Delta'(w_1)$ and $D(z_1)$ defined as before. Let c be the line segment from w_1 to b. Suppose for contradiction $b \in \Delta'(w_1)$, and recall that $b \notin \Delta'(w_0)$. We are done if we can show $a \in \Delta'(w_1)$. Note that all of c would be contained in $\Delta'(w_0) \cap \Delta'(w_1)$ except for the last point, b. Note that the inverse functions $f_0^{-1} : \Delta'(w_0) \to D(z_0)$, and $f_1^{-1} : \Delta'(w_1) \to D(z_0)$ have the same values on all of c except b. By continuity, it follows that $a \in \overline{D(z_1)}$. Since b is in the interior of $\Delta'(w_1)$, it follows that $a \in D(z_1)$. Thus, $R(w_1) \leq |w_1 - b|$. ■