Interpolation of holomorphic functions

For RUMA math mingle, 23 April, 2021.
**Definition**

Let $U \subseteq \mathbb{C}$ be open and connected. We say that $f : U \to \mathbb{C}$ is **holomorphic** if for all $z \in U$, the following limit exists:

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}.$$  

The set of all holomorphic functions on $U$ is denoted by $\mathcal{O}(U)$. 

Some extrinsic motivation as to why we care about $\mathcal{O}(U)$: applications to topology, analytic number theory, geometry, analysis, algebraic geometry, analytic combinatorics.
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Some extrinsic motivation as to why we care about $O(U)$: applications to topology, analytic number theory, geometry, analysis, algebraic geometry, analytic combinatorics.

In particular, we restrict to $\text{End}(\mathbb{D}) := O(\mathbb{D}, \mathbb{D})$, the space of holomorphic functions mapping the unit disk into itself.
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**Problem**

*Given initial data* $z_1, z_2, \ldots, z_n \in \mathbb{D}$, *and target data* $w_1, w_2, \ldots, w_n \in \mathbb{D}$, does there exist a holomorphic function $f \in \text{End}(\mathbb{D})$ so that $f(z_i) = w_i$ for all $1 \leq i \leq n$?
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Note that this problem is trivial in the case of \( \mathcal{O} (\mathbb{D}) \) as \( n + 1 \) many points in \( \mathbb{C} \) determine a unique polynomial of degree at most \( n \) (see the Vandermonde matrix).
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**Problem**

Given initial data \( z_1, z_2, \ldots, z_n \in \mathbb{D} \), and target data \( w_1, w_2, \ldots, w_n \in \mathbb{D} \), does there exist a holomorphic function \( f \in \text{End}(\mathbb{D}) \) so that \( f(z_i) = w_i \) for all \( 1 \leq i \leq n \)?

Note that this problem is trivial in the case of \( O(\mathbb{D}) \) as \( n + 1 \) many points in \( \mathbb{C} \) determine a unique polynomial of degree at most \( n \) (see the Vandermonde matrix).

Thus, we work with \( \mathbb{D} \) as otherwise the problem is too easy.
Example

Initial data: 0, 2/5, and target data: 1/2, 3/4.
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Such an $f$ exists by checking:

$$f(z) = \frac{z + 1/2}{1 + z/2}.$$
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Such an $f$ does not exist.
(Schwarz-Pick) Fix $f \in \text{End}(\mathbb{D})$. For all $z, w \in \mathbb{D}$,

$$\left| \frac{f(z) - f(w)}{1 - f(w)f(z)} \right| \leq \left| \frac{z - w}{1 - wz} \right|.$$
Theorem

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We see that,

$$\left| \frac{1/2 - 3/4}{1 - (1/2) \cdot (3/4)} \right| = 2/5 \text{ and } \left| \frac{0 - 1/5}{1 - (1/5) \cdot 0} \right| = 1/5.$$
For initial data $z_1, z_2$, and target data $w_1, w_2$ we know from the Schwarz-Pick theorem that if an $f \in \text{End}(\mathbb{D})$ satisfies the interpolation problem, we must have

$$\left| \frac{w_1 - w_2}{1 - \overline{w_2}w_1} \right| \leq \left| \frac{z_1 - z_2}{1 - \overline{z_2}z_1} \right|.$$
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This occurs if and only if the matrix,

$$\begin{pmatrix}
\frac{1-|w_1|^2}{1-|z_1|^2} & \frac{1-w_1\overline{w}_2}{1-z_1\overline{z}_2} \\
\frac{1-w_2\overline{w}_1}{1-\overline{z}_2 z_1} & \frac{1-|w_2|^2}{1-\overline{z}_2^2}
\end{pmatrix}$$

is positive semidefinite. That is, if and only if it has nonnegative determinant.
Theorem (Nevanlinna-Pick) Given initial data $z_1, z_2, \ldots, z_n$ and target data $w_1, w_2, \ldots, w_n$, there exists $f \in \text{End}(D)$ satisfying the data if and only if the matrix,
\[
\begin{pmatrix}
1 - w_j w_k & 1 - z_j z_k
\end{pmatrix}
\]
for $j, k = 1, \ldots, n$ is positive semidefinite.

Note that Sylvester's criterion says this is equivalent to the condition that all the determinants,
\[
D_m = \left| \begin{pmatrix}
1 - w_j w_k & 1 - z_j z_k
\end{pmatrix}
\right|
\]
are nonnegative for all $1 \leq m \leq n$.

The proof for Nevanlinna-Pick interpolation is constructive. Uniqueness occurs if and only if $D_n = 0$.

Generalizations of this problem are studied in modern research by operator theorists.

Has application in control theory (see Allen Tannenbaum wikipedia).

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August 23, 2021 8 / 8
Theorem

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\left( \frac{1-w_j w_k}{1-z_j z_k} \right)_{j,k=1}^n
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