1. Introduction

Consider \( \{ e_1, e_2, \ldots, e_n \} \), a fixed ordered basis of \( \mathbb{C}^n \). Recall that the standard flag of \( \mathbb{C}^n \) is defined by \( E_1 \subset E_2 \subset \cdots \subset E_n \), where \( E_k = \text{span}\{e_1, e_2, \ldots, e_k\} \). We now introduce partial flag varieties. Let \( m = (m_1, m_2, \ldots, m_k) \), where \( 0 < m_1 \leq m_2 \leq \cdots \leq m_k < n \). Let \( X = \text{Fl}(m, n) \), a partial flag variety, then \( X = \{ (V_{m_1} \subset V_{m_2} \subset \cdots \subset V_{m_k} \subseteq \mathbb{C}^n) \mid \dim(V_{m_i}) = m_i \} \). Let \( E_m = (E_{m_1} \subset E_{m_2} \subset \cdots \subset E_{m_k}) \in X \) and let \( P \) denote the stabilizer of \( E_m \), i.e. \( \{ g \in GL(n) \mid g.E_m = E_m \} \). Let \( m_0 = 0 \) and \( m_{k+1} = n \), then \( P \) is the group of invertible block upper triangular matrices, where the dimension of the \( i \)th block is \( m_{i+1} - m_i \). The following are other key subgroups of \( GL(n) \) that we will consider: \( T \) is the torus and the set of invertible diagonal matrices, \( B \) is the Borel subgroup and is the set of invertible upper triangular matrices. We have the following relation among these subgroups: \( T \subseteq B \subseteq P \subseteq GL(n) \). We will also only consider Weyl groups of type \( A \) so \( W = S_n \).

2. Exercises

Exercise 1. We shall show that \( X^T = \{ w.E_m \mid w \in S_n \} \).

It is easy to see that \( w.E_m \in X^T \). Now to show the other direction, consider \( t \in T \) and write \( t \) as

\[
\begin{bmatrix}
\lambda_1 & 0 & 0 & \ldots & 0 \\
0 & \lambda_2 & 0 & \ldots & 0 \\
0 & 0 & \lambda_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \lambda_n
\end{bmatrix}
\]

Let \( M \in X \) and write \( M \) as \( [v_1 \ v_2 \ \ldots \ v_n] \) where \( v_i \) is a \( n \times 1 \) vector. Then in order for \( M \) to be in \( X \), we must have that the span of the first \( m_1 \) vectors after multiplying by scalars from \( t \) must be the same as \( E_{m_1} \), so it follows that there can only be \( m_1 \) nonzero rows among these first \( m_1 \) vectors because we allow \( t \) to vary. Thus using column operations, we can get that these \( m_1 \) vectors only have \( m_1 \) nonzero entries combined and each is in a different row. We continue this process for \( E_{m_2} \) all the way to \( E_{m_k} \) and \( \mathbb{C}^n \) from which we can see that \( M \) has the form \( w.E_m \) for some \( w \in S_n \).

Normalizer. Define \( N_G(T) = \{ g \in GL(n) \mid gTg^{-1} = T \} \). One can show that \( N_G(T) = \{ \text{all permutations matrices with arbitrary nonzero numbers in the 1's places} \} = S_nT \). We now show some relations between \( N_G(T) \) and \( W \).

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Exercise 2. \( W = S_n = N_G(T)/T \) and \( W_P = S_n \cap P = N_P(T)/T \) and furthermore \( W_P \leq W \).

First proof that \( N_G(T) = S_n.T \). Let \( M \in N_G(T) \) and let \( T' \) be a diagonal matrix such that \( T_{ii} = i \), then \( MT'M^{-1} = T^* \) is also a diagonal matrix. Since conjugation preserves spectrum, it must be that \( T_{ii} = \sigma(i) \) for some permutation \( \sigma \). Since \( MT = T'M, S_{ij}j = \sigma(i)S_{ij}, S_{ij}(j - \sigma(i)) = 0 \). So \( S_{ij} = 0 \) for all \( \sigma(i) \neq j \). Each row has only one non-zero entry. So \( M \) must be of the form \( \sigma \).

Now since \( N_G(T) = S_n.T \), the homomorphism \( \phi : N_G(T) \to W \) where \( \phi(M) = w \) with \( M = wT' \) for some permutation \( w \) and diagonal matrix \( T' \). \( w \) is the identity permutation if and only if \( M \in T \). So the kernel of \( \phi \) is \( T \). Therefore \( N_G(T)/T \cong W \). Therefore we have our desired isomorphism and a similar argument works for \( W_P \).

Note that \( W_P \) is made up of the permutations in \( S_n \) that fit the shape of \( P \). It is clear from the definition of \( P \) that the identity element is in \( W_P \). If you consider the transpositions that generate all the possible permutations in one specific block of \( P \), then it is easy to see that \( W_P \leq W \) since these transpositions generate \( W_P \).

Exercise 3. We will show that for \( w \in W \) there exists a unique permutation \( u \in wW_P \) of minimal length where length of \( u = l(u) = \#\{(i < j) \mid u(i) > u(j)\} \).

We provide the following algorithm for constructing \( u \). Start with \( w \) and multiply \( w \) by \( s \in W_P \) only if \( l(ws) < l(w) \). Do this for all elements \( s \in W_P \) so that at the end, we have \( u = ws_{s_1}s_{s_2}\cdots s_{s_k} \). The claim is that this \( u \) is of minimal length. For sake of contradiction, suppose we have \( v \in W_P \) such that \( l(v) < l(w) \).

Then this means that there exists \( s \in W_P \) such that \( vs = u \) so \( v = us^{-1} \). Therefore \( l(us^{-1}) < l(u) \), however this is not possible by construction of \( u \). \( u \) is also unique. Suppose \( u_1 \) and \( u_2 \) both have minimal length. Then \( u_1 = u_2w_2 \) and \( u_2 = u_1w_1 \) for \( u_1, w_1 \in W_P \). Hence \( l(u_1) \leq l(u_1w_1) = l(u_2) \) and \( l(u_2) \leq l(u_2w_2) = l(u_1) \), by definition of \( u_1 \) and \( u_2 \) being elements of minimal length in \( wW_P \). We also have \( l(u_1) = l(u_2) \) so it follows that \( w_1 = w_2 = e \) and therefore, \( u_1 = u_2 \).

Grassmannian. Let \( X = Gr(m, n) \). We will view this Grassmannian as a partial flag variety. \( P \) consists of invertible block upper triangular matrices with two blocks. The block in the top left has size \( m \) and the block in the bottom right has size \( n - m \). We now describe \( W_P \) in terms of its generators. The generators are \{\{(i) \mid 2 \leq i \leq m\} \cup \{((m+1)j) \mid m+2 \leq j \leq n\} \}. Let \( W_P \subseteq W \) be the set of all such \( u \) described in exercise 3. In \( X \), \( W_P \) is the set of all permutations that send \((12\ldots m) \) to \( \lambda \) numbers that are ordered from lowest to highest so there are \( \binom{n}{m} \) elements in \( W_P \). Therefore we can discuss a bijection between young diagrams and \( W_P \). Note that we already have a bijection between young diagrams and \( X^T \) since \( X^T \) corresponds to Schubert symbols, which are used to construct a young diagram.

Exercise 4. We shall find a bijection between \( X^T \) and \( W_P \) and between young diagrams and \( X^T \).

We will show a bijection between \( W/W_P \) and \( X^T \) since \( W/W_P \) is essentially equivalent to \( W_P \). Let \( w.E_m \in X^T \) be an arbitrary element. We prove that \( \phi(w.E_m) = wW_P \) is a bijection. By exercise three, we know there exists a unique minimal length \( u \) for \( wW_p \), since \( uW_p = wW_p \) for some \( v \in W_p \). Now by definition of stabilizer \( v.E_m = E_m \) for all \( v \in W_P \). So \( \phi(v.E_m) = \phi(v.w.E_m) = \phi(u.E_m) = uW_P \). Since such a \( u \) exist and is unique. \( \phi \) is a bijection. A bijection with young diagrams follows through bijection between young diagrams and \( X^T \).

Note that by this bijection, \( l(u) = |\lambda| \). Cool! In fact, we can say more about the connection between \( u \) and \( \lambda \). For \( 1 \leq i \leq m \), the number of boxes in row \( i \) corresponds with the number of inversions with \( i \), i.e. the number of boxes = \#\{\( j \mid j > i \) and \( u(i) > u(j) \}\}. Therefore, \( u \) will completely determine \( \lambda \).