PERFECTLY MATCHED LAYERS AND THE ART OF CLOAKING

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Outline

I. Finite-Difference Time-Domain Problems
II. Posing the Open Boundary Problem
III. Perfectly Matched Layers
IV. Current Research in Cloaking
Finite-Difference Time-Domain (FDTD)

- Consider one dimensional wave equation problem with Dirichlet boundary conditions:

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \cdot \frac{1}{c^2} \quad \text{for} \ 0 \leq t \leq T, \ 0 \leq x \leq \ell
\]

\[
u(0; x) = f(x) \quad \frac{\partial u(0; x)}{\partial t} = g(x)
\]

\[
u(t; 0) = \Psi_1(t) \quad u(t; \ell) = \Psi_2(t).
\]

- The FDTD approach is to discretize the region into a mesh of \(n\) by \(m\) cells for which we hope

\[
U_{i,j} \approx u(\Delta t \cdot i; \Delta x \cdot j)
\]

for \(i = 0, \ldots, n - 1, j = 0, \ldots, m - 1, \Delta t = T/n, \) and \(\Delta x = \ell/m\)
Finite-Difference Time-Domain (FDTD)

• From our problem, we can derive

\[ U_{0,j} = f(\Delta x \cdot j),\ U_{i,0} = \Psi_1(\Delta t \cdot i),\ U_{i,m-1} = \Psi_2(\Delta t \cdot i),\]

\[ U_{1,j} = g(\Delta x \cdot j)\Delta t + \frac{s}{2} \cdot f(\Delta x \cdot (j + 1)) + \]

\[ (1 - s) \cdot f(\Delta x \cdot j) + \frac{s}{2} \cdot f(\Delta x \cdot (j - 1)) \]

and

\[ U_{i+1,j} = s \cdot U_{i,j+1} + 2 \cdot (1 - s) \cdot U_{i,j} + s \cdot U_{i,j-1} - U_{i-1,j} \]

for \( s = \left(\frac{c \Delta t}{\Delta x}\right)^2 \).

• These equations fully determine the mesh \( U_{i,j} \).
Plots of the two Dimension Solution
Posing the Problem

- Again considering the wave equation for a density function $u(t; \vec{x})$ on some bounded domain $\Omega$
  \[ \nabla^2 u = \frac{\ddot{u}}{c^2} \]
- Now, ask the deceptively hard question:
  “Is it possible to trick the solution into behaving on the interior of $\Omega$ exactly like the domain extends without bound?”
- In one dimension,
  “Yes, using absorbing boundary conditions (ABC)”
- In multiple dimensions,
  “Simple extensions of ABCs seem not to suffice”
Perfectly Matched Layers (PML)

• In a 1994 Paper, Jean-Pierre Berenger outlines method for an absorbing layer, adjacent to the boundary, which applies an exponential decay to incident and departing waves.

• There are many formulations of methods Berenger developed, but the process can be described as a coordinate transform:

\[
\frac{\partial}{\partial x} \rightarrow \frac{1}{1 + i \frac{\sigma(x)}{\omega}} \frac{\partial}{\partial x}
\]

where \( \sigma > 0 \) in the abortion layer and \( \sigma = 0 \) otherwise.
Current Research in Cloaking

- Water ripples, sound, light, and electromagnetic radiation are all waves.

The above images show traveling waves on an elastic surface containing a hole -- with and without a local absorbing layer.

Image from N. Stenger, et al. (2012)
References
