Compactness in the Mathematical Universe

Ava Ostrem

July 16, 2025

Compactness

Compactness is a phenomenon in mathematics where structures are determined by their local behavior.

Compactness in nature: fractals

Compactness in mathematics

Godel's compactness

A set of sentences in a first-order language has a model if and only if each finite subset has a model.

Compactness in mathematics

Godel's compactness

A set of sentences in a first-order language has a model if and only if each finite subset has a model.

Topological compactness

A topological space is compact if every open cover of the space has a finite subcover.

REU Goals

- Finding other forms of compactness for large cardinals.

REU Goals

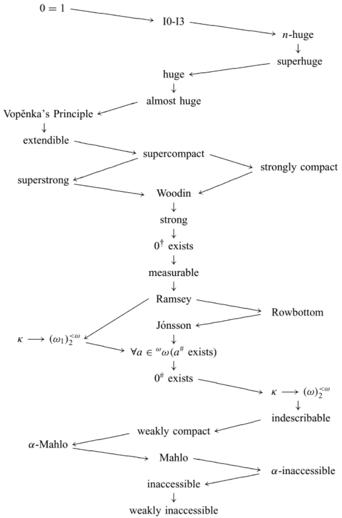
- Finding other forms of compactness for large cardinals.
- Finding other forms of compactness for different algebraic structures.

Large cardinal axioms

- Large cardinals are infinite sets which are so 'large' that their existence is not provable in ZFC.
- The existence of a large cardinal can be added to the theory of ZFC as an axiom.

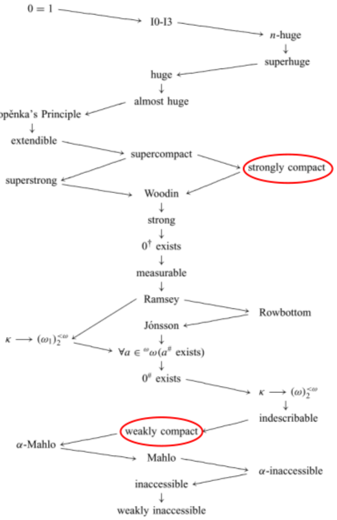
Large cardinal chart

The arrows indicates direct implications or relative consistency implications, often both.



Large cardinal chart

The arrows indicates direct implications or relative consistency implications, often both.



Compactness for free abelian groups

A **free** abelian group is a direct sum of copies of $\mathbb{Z}$.

Theorem (Folklore)

Let κ be a weakly compact cardinal. If G is a κ-free group of cardinality less than or equal to κ, then G is free.

Compactness for Σ-cyclic groups

A group is Σ-**cyclic** if it is the direct sum of cyclic groups.

Compactness for Σ-cyclic groups

A group is Σ-**cyclic** if it is the direct sum of cyclic groups.

Theorem (Calderoni-O.)

Let κ be a weakly compact cardinal. If G is a κ-Σ-cyclic group of cardinality less than or equal to κ, then G is Σ-cyclic.

Compactness for Σ-cyclic groups

A group is Σ-**cyclic** if it is the direct sum of cyclic groups.

Theorem (Calderoni-O.)

Let κ be a weakly compact cardinal. If G is a κ-Σ-cyclic group of cardinality less than or equal to κ, then G is Σ-cyclic.

The proof uses filtrations and stationary reflection.

Strong compactness for free abelian groups

A **free** abelian group is a direct sum of copies of $\mathbb{Z}$.

Theorem

Let κ be an ω_1-strongly compact cardinal. If G is a κ-free group, then G is free.

Strong compactness for free abelian groups

A **free** abelian group is a direct sum of copies of $\mathbb{Z}$.

Theorem

Let κ be an ω_1-strongly compact cardinal. If G is a κ-free group, then G is free.

See Eklof and Mekler, 2002 for the proof.

Strong compactness for Σ-cyclic groups

Theorem (Calderoni-O.)

Let κ be an ω_1-strongly compact cardinal. If G is a κ-Σ-cyclic group, then G is Σ-cyclic.

The proof uses ultraproducts.

Acknowledgments

- Thank you to my mentor Dr. Filippo Calderoni for his guidance and support!
- Thank you to the DIMACS REU for hosting this research project.
- This work is supported by the Rutgers Department of Mathematics and NSF Grant DMS #2348819.

Bibliography

Eklof, Paul C. and Alan H. Mekler (2002). *Almost free modules*. Revised. Vol. 65. North-Holland Mathematical Library. Set-theoretic methods. North-Holland Publishing Co., Amsterdam, pp. xxii+597. ISBN: 0-444-50492-3.

Kanamori, Akihiro (1994). *The higher infinite*. Perspectives in Mathematical Logic. Large cardinals in set theory from their beginnings. Springer-Verlag, Berlin, pp. xxiv+536. ISBN: 3-540-57071-3.