The Pinning Number of Overlapping Rectangles

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The Problem

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How many pins do we need?
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- It is enough to consider the case of rectangles with independence $\alpha$ on $\alpha \times \alpha$ grid.
- If $k\alpha$ pins suffice, then $k \geq 2$. 
Crossings are Bad

Theorem

There is $k$ such that $k^\alpha$ pins suffice to pin down any set of rectangles that does not contain two that intersect in this way.

We will omit the proof which is a bit technical.
Reduction to $\alpha \times \alpha$ Grid

Theorem

If we can pin down any set of independence $\alpha$ on $\alpha \times \alpha$ grid (see figure) with $k\alpha$ pins, then we can pin down any set of independence $\alpha$ by at most $9k\alpha$ pins.
Asymptotics

Theorem

If $k^\alpha$ pins always suffice to pin down any set of independence $\alpha$, then $k \geq 2$. 

Sketch of a proof:

For any given $n$ we will construct a set of independence at most $n + 2$ for which we need at least $2n$ pins.

For $n$ big it is $\frac{2n}{n+2} \rightarrow 2$. 

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For $n$ big it is $\frac{2n}{n+2} \to 2$. 
Since there are $4n$ rectangles and each point lies in at most $2$ rectangles we need at least $2n$ pins.

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It can be shown that this set has (for $n \geq 3$) independence number $n + 2$ and we need at least $2n + 2$ pins.
Conclusions

While we have not managed to solve the general problem, we have at least simplified it.
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The fact that for many configurations we need only linearly many pins hints that the number of pins needed is indeed linear in $\alpha$. 
Thanks

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Thank you for your attention.