Computations of the involutive concordance invariants of (1,1)-knots

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What is a Knot?

- A *knot* is an embedding $S^1 \hookrightarrow S^3 = \mathbb{R}^3 \cup \infty$.
- A *link* is an embedding of a disjoint union $S^1 \cup \cdots \cup S^1 \hookrightarrow S^3$.

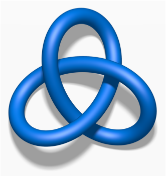


Figure: Trefoil Knot

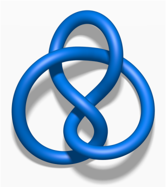


Figure: Figure-Eight Knot

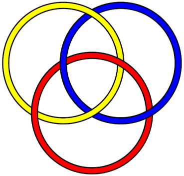


Figure: Borromean Rings: a 3-component link

The source of the images is the Knot Atlas: http://katlas.org/wiki/

Special Types of Knots

Torus knots:

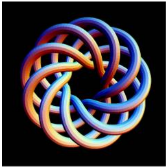


Figure: T(9,5)

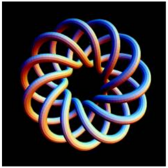


Figure: T(11,4)

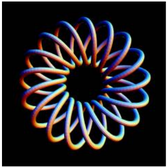


Figure: T(17,3)

Pretzel Knots:

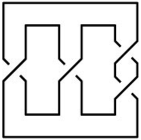


Figure: 4_1

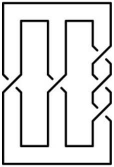


Figure: 5_2

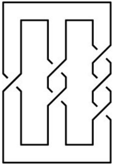


Figure: 6_2

The source of the images is the Knot Atlas: http://katlas.org/wiki/

Why Knots are Important

- Many things in the real world are knotted - Applications in studying DNA
- Knots are an early case of the embedding problem.
- Knots are very closely related to 3- and 4-dimensional manifolds.

Theorem

(Lickorish, Wallace): Every closed 3-dimensional manifold can be described in terms of some link and an integer associated to each component.

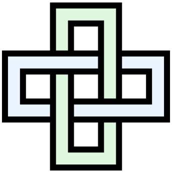


Figure: Solomon's Knot Square: a 2-component link

The source of the image is the Knot Atlas: http://katlas.org/wiki/

Knot Concordance

Definition

Two knots K_1 and K_2 are *concordant* if they cobound a smooth, properly embedded cylinder in $S^3 \times [0, 1]$.

- Concordance is an equivalence relation.
- We can study *concordance invariants* for knots.

Definition

A knot is considered *slice* if it is concordant to the unknot.

Knot Concordance

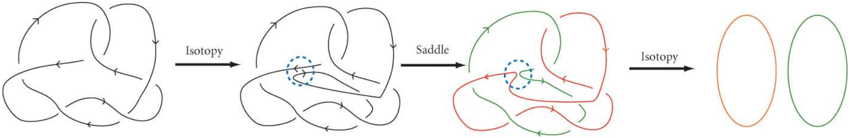


Figure: The Stevedore Knot is slice.

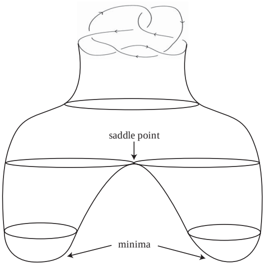


Figure: Another representation of the sequence above

The source of the images is *Slice Knots: Knot Theory in the 4th Dimension* by Peter Teichner (2010).

The Chain Complex CFK^∞

To each knot K we can associate the complex $CFK^\infty(K)$ which contains extensive geometric information about the knot.

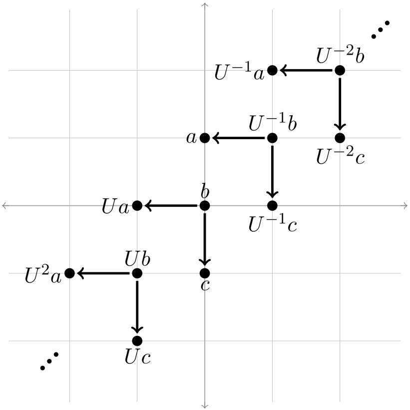


Figure: $CFK^\infty(K)$ for the right-handed trefoil knot

Source of the figure: *A Survey on Heegaard Floer Homology and Concordance* by Jennifer Hom (2017).

The Automorphism ι_K

Definition

$\iota_K : CFK^\infty(K) \to CFK^\infty(K)$

- An automorphism on $CFK^\infty(K)$ that preserves the structure of the complex.
- Usually close to a reflection over the line $i = j$.
- Contains interesting 4-dimensional data.
- Can detect the fact that the figure-eight knot isn't slice.

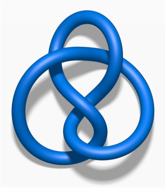


Figure: The figure-eight knot

The source of the image is the Knot Atlas: http://katlas.org/wiki/

What Knots We Considered

ι_k has been computed for

- Torus knots
- Alternating knots
- Some pretzel knots (previous REU)

We computed ι_k and the corresponding involutive concordance invariants for

- (1,1)-knots (for which ι_k hadn't been computed yet)

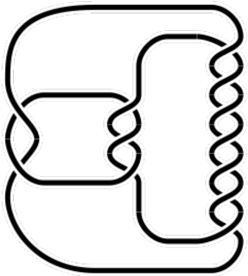


Figure: The pretzel knot $P(-2, 3, 7)$

The source of the image is: https://wikipedia.org/wiki/(-2,3,7)_pretzel_knot

Example: 10_{161}

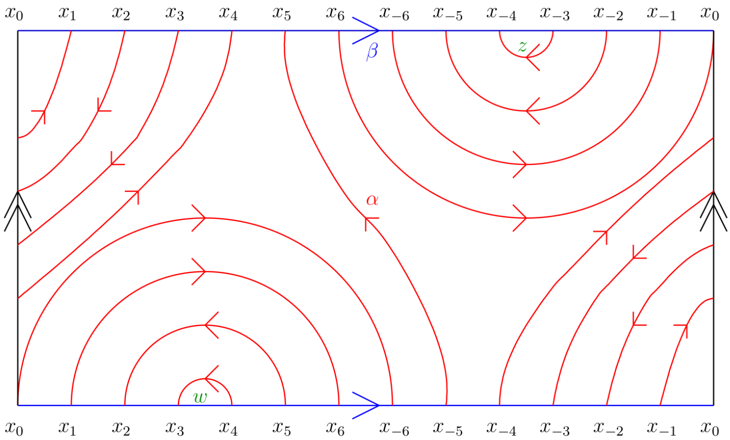


Figure: The knot 10_{161} represented by the 4-tuple $(6, 4, -3, -1)$

Figure source: *Geometry of (1, 1)-Knots and Knot Floer Homology* by Racz

Example: CFK^∞ for 10_{161}

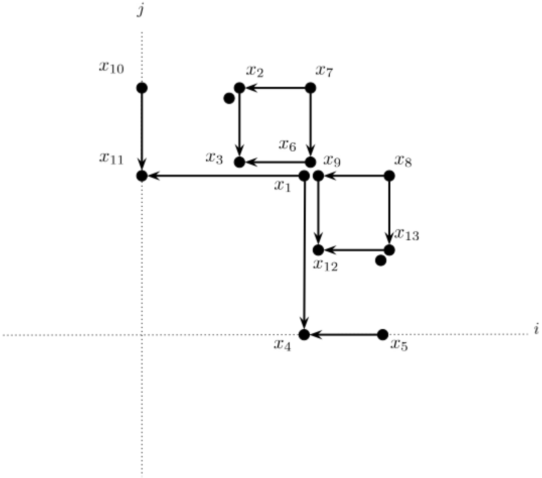


Figure: CFK^∞ for 10_{161}.

V_0 can be easily computed in an algorithmic way.

Definition

We define the chain complex A_0^- by

$$A_0^- = C\{(i,j) : i,j \leq 0\}.$$

Definition

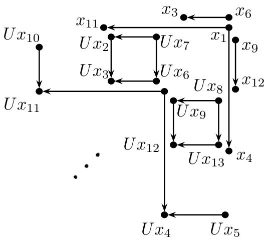
The concordance invariant V_0 is given by

$$V_0 = -\frac{1}{2} \max \{r : \exists x \in H_r(A_0^-) \text{ such that } U^n x \neq 0 \text{ for all n } \}.$$

Computing V_0 for 10_{161}

We find the grading of the topmost element in the tower built from the homology of the complex.

A_0^-



$[x_{11}]$

$$[Ux_1 + Ux_5 + Ux_{10}]$$
$$\downarrow$$
$$[U^2x_1 + U^2x_5 + U^2x_{10}]$$
$$\downarrow$$
$$[U^3x_1 + U^3x_5 + U^3x_{10}]$$
$$\downarrow$$
$$\vdots$$

Results: $V_0(10_{161}) = 0$, $V_0(\overline{10_{161}}) = 1$

The Involutive Concordance Invariants $\underline{V}_0$ and $\overline{V}_0$

$\underline{V}_0$ and $\overline{V}_0$ are in general more difficult to compute than V_0, because they first require the computation of ι_K.

Definition

Let AI_0^- be the mapping cone $\text{Cone}(A_0^- \xrightarrow{\iota_K + \mathrm{Id}} QA_0^-)$. Then, the involutive concordance invariants $\underline{V}_0$ and $\overline{V}_0$ are given by:

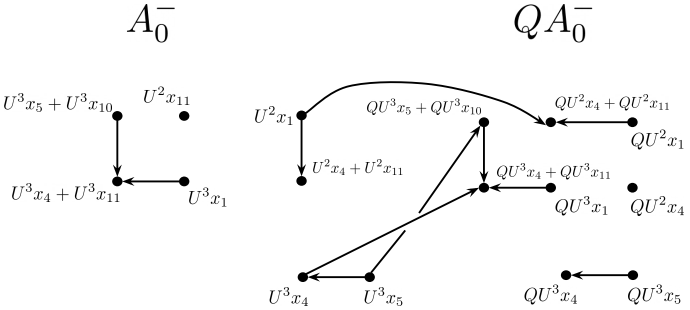
$$\underline{V}_0 = -\frac{1}{2}\left(\max\{r : \exists x \in H_r(AI_0^-) \text{ s.t. } U^n x \neq 0 \text{ and } U^n x \notin \mathrm{Im}(Q) \quad \forall n\} - 1\right).$$

$$\overline{V}_0 = -\frac{1}{2}\max\{r : \exists x \in H_r(AI_0^-) \text{ s.t. } U^n x \neq 0 \quad \forall n, \exists m \geq 0 \text{ s.t. } U^m x \in \mathrm{Im}(Q)\}.$$

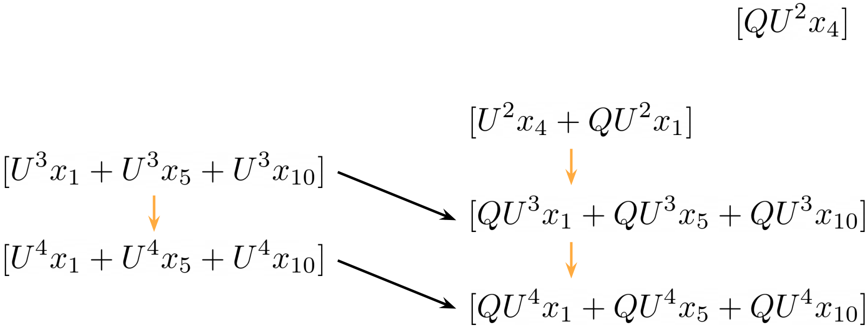
Im(Q) denotes the image of Q.

Computing the Involutive Concordance Invariants for 10_{161}

The mapping cone $\mathrm{Cone}(A_0^- \xrightarrow{\iota_K+\mathrm{Id}} QA_0^-)$ is shown below:



We find the grading of the topmost element of each of the two towers built from the homology of the mapping cone.



$\underline{V}_0(10_{161})$	$\overline{V}_0(10_{161})$	$\underline{V}_0(\overline{10_{161}})$	$\overline{V}_0(\overline{10_{161}})$
0	-1	1	1

Results Table

K	$V_0(K)$	$\underline{V}_0(K)$	$\overline{V}_0(K)$	$V_0(\overline{K})$	$\underline{V}_0(\overline{K})$	$\overline{V}_0(\overline{K})$
10_{128}	1	1	1	0	0	-1
10_{132}	0	0	-1	1	1	1
10_{136}	0	0	0	0	0	0
10_{139}	2	2	1	0	0	-2
10_{145}	0	0	-1	1	1	1
$11n_{12}$	1	1	1	0	0	-1
$11n_{19}$	1	1	1	0	0	-1
$11n_{20}$	0	0	0	0	0	0
$11n_{38}$	0	0	-1	1	1	1
$11n_{57}$	1	2	1	0	0	-1
$11n_{61}$	1	1	0	0	0	0
$11n_{79}$	0	0	0	0	0	0
$11n_{96}$	0	0	-2	2	2	2

Further Results

- We verified the diagrams for the 26 11- and 12-crossing (1,1)-knots listed in *Geometry of (1,1)-Knots and Knot Floer Homology* by Racz.
- We identified three which were incorrect.
- Rasmussen's *Knot Polynomials and Knot Homologies* contained a correction for $12n_{749}$.
- We corrected $12n_{404}$ by narrowing down the possibilities based on some knot invariants.

A Unique Example

An example of one of the most complicated complexes we found:

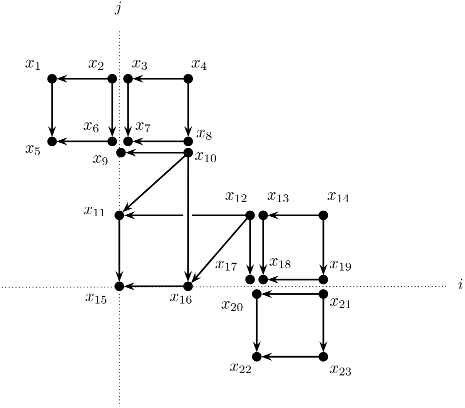


Figure: $CFK^\infty(11n_{61})$

Acknowledgements

- Thank you to our mentors Dr. Hendricks and Karuna Sangam!
- Additional thanks to the DIMACS REU at Rutgers University for providing this research opportunity!
- This research was funded by NSF CAREER grant DMS-2019396.

- Thank you for listening!