# Morphisms

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# Definition of Morphism

- Goal: want to find the natural maps between algebraic varieties.
- Let $k$ be an algebraically closed field and $X \subset k^n$, $Y \subset k^m$ be algebraic varieties.
- A *morphism* $F : X \to Y$ is a map given by

$$F : (a_1, \cdots, a_n) \mapsto (f_1(a_1, \cdots, a_n), \cdots, f_m(a_1, \cdots, a_n)).$$

- This is a restiction of polynomial maps from $k^n$ to $k^m$.

# A map of $k$-algebras

- Now, we use the same polynomials $f_i$ to define a map of rings

$$F^{\#} : k[y_1, \cdots, y_m] \to k[x_1, \cdots, x_n]$$

  sending $y_i$ to $f_i(x_1, \cdots, x_n)$.

- To say that $F$ restricts to a map carrying $X$ to $Y$ means that if $g \in I(Y)$, then

$$F^{\#}(g) = g(f_1(a_1, \cdots, a_n), \cdots, f_m(a_1, \cdots, a_n)) \in I(X).$$

- Thus, $F^{\#}$ induces a map of $k$-algebras

$$F^{\#} : A(Y) = k[y_1, \cdots, y_m]/I(Y) \to k[x_1, \cdots, x_n]/I(X) = A(X).$$

- Two maps with the same restriction to $X$ induce the same map from $A(X) \to A(Y)$.

# Example

- Consider the morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ given by
$$(s, t) \mapsto (s^3, s^2t + st^2, t^3)$$
where $(s, t)$ has homogeneous coordinates.
- How can we describe the image of $\varphi$?
- Let the homogeneous coordinates in $\mathbb{P}^2$ by $x, y, z$.
- Define $I$ to be the ideal of all homogeneous forms $F(x, y, z)$ that vanish identically when $s^3, s^2t + st^2, t^3$ are substituted for $x, y, z$.
- The algebraic set defined by $I$ is the smallest set that contains the image of $\varphi$.

# Example Continued

- We want to eliminate $s$ and $t$ from the forms

$$x - s^3, y - (s^2 t + st^2), z - t^3.$$

- Essentially by taking the intersection of the ideal $(x - s^3, y - (s^2 t + st^2), z - t^3)$ with the polynomial ring $k[x, y, z]$.
- This yields the ideal generated by

$$F = y^3 - x^2 z - 3xyz - xz^2.$$

- Thus, the image of $\varphi$ is defined by $F$.

# Morphism in the Context of Rational Functions

- Consider an algebraic variety $V \subset \mathbb{C}^n$.
- A rational function $f : V \to \mathbb{C}^n$ is called *regular* at $v \in V$ if there are polynomials $g$ and $h$ such that $f = g/h$ and $h(v) \neq 0$.
- $f$ is called *regular on* $V$ if it is regular at each point.

# Example of Regular Function

- Let $V = (x^2 - y^3 = 0) \subset \mathbb{C}^2$ and let $f$ be the rational function $f = x/y$ restricted to $V$.
- $f$ is regular everywhere but at $(0, 0)$.
- Assume that $x/y = a(x, y)/b(x, y)$ and $b(0, 0) \neq 0$.
- Then, $xb(x, y) - ya(x, y) = 0$ on $V$, hence is divisible by $x^2 - y^3$ by the Nullstellensatz.
- $b(x, y)$ has a nonzero constant term, so the coefficient of $x$ in $xb(x, y) - ya(x, y)$ is not 0.
- Thus, $x^2 - y^3$ does not divide $xb(x, y) - ya(x, y)$, which is a contradiction.
- Interestingly, $f^2 = x^2/y^2 = y$ restricted to $V$ is regular.

# Normal

- $V$ is said to be *normal at* $v \in V$ if every rational function bounded in some neighborhood of $v$ is regular at $v$.
- $V$ is called *normal* if it is normal at every point.
- If $V$ is normal at $v$, then a rational function is regular at $v$ if and only if it is continuous at $v$.
- Prop: Let $C$ be an algebraic curve. Then $C$ is normal iff it is smooth.
- Cor: Let $V$ be a normal variety. Then,

$$\dim \mathrm{Sing} V \leq \dim V - 2.$$

# Some More Facts

- Hartogs' Theorem:
  - Let $V$ be a normal variety and let $W \subset V$ be a subvariety such that $\dim W \leq \dim V - 2$.
  - Let $f$ be a regular function on $V - W$.
  - Then, $f$ extends to a regular function on $V$.
- Prop: Let $V$ be an irreducible projective variety and $f$ a regular function on $V$. Then, $f$ is constant.

# Definition of Map

- Pick rational functions $f_1, \cdots, f_n$ on $V$.
- A map $F : V \to \mathbb{C}^n$ is given by

$$v \mapsto (f_1(v) : \cdots : f_n(v) : 1).$$

- $F$ is defined at $v$ iff there is a $g$, regular at $v$ such that all the $f_i g$ are regular at $v$ and $(f_1 g : \cdots : f_n g : g)$ is not identically zero at $v$.
- Dimension formula:
  - Let $f : V \to W$ be a regular map.
  - Then, for any $w \in W, f^{-1}(w)$ is either empty or

$$\dim f^{-1} \geq \dim V - \dim W.$$

# Another Definition

- A point in $\mathbb{CP}^n$ is given uniquely by a line in $\mathbb{C}^{n+1}$.
- If $f : V \to \mathbb{CP}^n$ is map, then this associates to each $v \in V$ a line in $\mathbb{C}^{n+1}$.
- So, the map can be described as a subset

  $$L \subset V \times C^{n+1}$$

  such that for each $v \in V, L \cap (\{v\} \times \mathbb{C}^{n+1})$ is a line $L_v$, and this line "varies algebraically" with $v$.
- Conversely, any such subset $L$ defines a map into $\mathbb{CP}^n$.

# Sources


- Eisenbud, David: *Commutative Algebra with a View Toward Algebraic Geometry*
- Kollar, Janos: "The structure of algebraic threefolds: an introduction to Mori's program" https://projecteuclid.org/download/pdf_1/euclid.bams/1183554173