# Hartshorne's *Algebraic Geometry*: Varieties

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# Some Initial Definitions

- Let $k$ be a fixed algebraically closed field
- An affine n-space over $k$, denoted $\boldsymbol{A}^n$, is the set of all n-tuples of elements of $k$.
- $P = (a_1, \cdots, a_n) \in \boldsymbol{A}^n$ with $a_i \in k$ is called a point, and the $a_i$ are called the coordinates of P.
- $A = k[x_1, \cdots, x_n]$ is the polynomial ring in $n$ variables over $k$. We interpret an element $f \in A$ as a function $\boldsymbol{A}^n \to k$.
- Given $T \subseteq A$, the zero set of T is

$$Z(T) = \{P \in \boldsymbol{A}^n : f(P) = 0 \text{ for all } f \in T\}.$$

# Algebraic Sets and Zariski Topology

- A subset $Y \subseteq \boldsymbol{A}^n$ is an algebraic set if there exists a subset $T \subseteq A$ such that $Y = Z(T)$.
- Proposition 1.1:
  1. The union of two algebraic sets is an algebraic set.
  2. The intersection of any family of algebraic subsets is an algebraic set.
  3. The empty set and whole space are algebraic sets.
- Define the Zariski Topology on $\boldsymbol{A}^n$ by taking the open subsets to be the complements of the algebraic sets. By the proposition, this is a topology.

# Irreducibility

- A nonempty subset $Y$ of a topological space $X$ is irreducible if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two closed, proper subsets of Y.
- Example: the affine line $\boldsymbol{A}^1$ is irreducible
  - Every ideal in $A = k[x]$ is principle (can show using the remainder theorem), so every algebraic set is the set of zeros of a single polynomial.
  - Since $k$ is algebraically closed, every nonzero polynomial can be written as $f(x) = c(x - a_1) \cdots (x - a_n)$ with $c, a_1, \cdots, a_n \in k$.
  - Thus, $Z(f) = \{a_1, \cdots, a_n\}$.
  - $\boldsymbol{A}^1$ is irreducible, because its only proper closed subsets are finite, yet it is infinite (since $k$ is infinite).
- Now we can define the affine variety as an irreducible closed subset of $\boldsymbol{A}^n$ (with the induced topology).

# Ideals and Zero Sets

- For a subset $Y \subseteq \boldsymbol{A}^n$, the ideal of Y in A is

$$I(Y) = \{f \in A : f(P) = 0 \text{ for all } P \in Y\}.$$

- Proposition 1.2:
  1. If $T_1 \subseteq T_2$ are subsets of $A$, then $Z(T_1) \supseteq Z(T_2)$.
  2. If $Y_1 \subseteq Y_2$ are subsets of $\boldsymbol{A}^n$, then $I(Y_1) \supseteq I(Y_2)$.
  3. For any two subsets $Y_1, Y_2$ of $\boldsymbol{A}^n$, we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
  4. For any ideal $a \subseteq A, I(Z(a)) = \sqrt{a}$, the radical of $a$, defined as

$$\sqrt{a} = \{f \in A : f^r \in a \text{ for some } r > 0\}.$$

  5. For any subset $Y \subseteq \boldsymbol{A}^n, Z(I(Y)) = \overline{Y}$, the closure of $Y$, defined as the intersection of all closed sets containing $Y$.

# Main Ideas and the Affine Coordinate Ring

- Corollary 1.4
  1. There is a one-to-one correspondence between algebraic sets in $\boldsymbol{A}^n$ and radical ideals in $A$.
  2. This is given by $Y \mapsto I(Y)$ and $a \mapsto Z(a)$.
  3. An algebraic set is irreducible $\iff$ its ideal is a prime ideal.
- If $Y \subseteq \boldsymbol{A}^n$ is an affine algebraic set, the <u>affine coordinate ring of $Y$</u> is $A(Y) = A/I(Y)$.
- Remark: If Y is an affine variety, then $A(Y)$ is an integral domain and a finitely generated $k$-algebra.