The Genus of a Curve

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Genus

- The most trivial curve is $\mathbb{P}^1$, which is the sphere S^2.
- Broadly, the genus of a curve is the number of handles added to a sphere.
  - A sphere has genus $g = 0$.
  - A torus has genus $g = 1$.
- Formally, given a curve X we can define the *arithmetic genus* $p_a(X)$ or the *geometric genus* $p_g(X)$.
- The two definitions are equivalent, and I will just focus on the arithmetic genus for now.

Hilbert Function and Polynomial

- Let k be an algebraically closed field.
- Let M be a finitely generated graded module over $k[x_1, \cdots, x_r]$, with grading by degree.
- That is, $M = \oplus_{s \in \mathbb{Z}} M_s$. Note that M is a k-vector space.
- We define the *Hilbert function* $h_M : \mathbb{N} \to \mathbb{N}$ by

$$h_M(s) = \dim_k(M_s).$$

- Hilbert-Serre Theorem: $h_M(s)$ agrees, for large s, with a polynomial of degree $\leq r$.
- We call this polynomial the *Hilbert polynomial of M*.

Proof of Hilbert-Serre

- Theorem: If M is a finitely generated graded module over $k[x_0, \cdots, x_r]$, then $h_M(s)$ agrees, for large s, with a polynomial of degree $\leq r$.
- Lemma:
  - Let $h(s) \in \mathbb{Z}$ be defined for all natural numbers s.
  - Define the *first difference* $h'(s) = h(s) - h(s-1)$.
  - If the first difference agrees with a polynomial of degree $\leq n-1$ having rational coefficients for $s \geq s_0$, then $h(s)$ agrees with a polynomial of degree $\leq n$ having rational coefficients for all $s \geq s_0$.

Proof of Lemma

- Suppose that $Q(s)$ is a polynomial of degree $\leq n-1$ with rational coefficients such that $h'(s) = Q(s)$ for $s \geq s_0$.
- Now define $P(s)$ as follows:
  - For $s \geq s_0$, set $P(s) = h(s)$.
  - For $s \leq s_0$ set $P(s) = h(s_0) - \sum_{t=s+1}^{s_0} Q(t)$.
- Then, $P(s) - P(s-1) = Q(s)$ for all integers s.
- $P(s)$ is a polynomial of degree $\leq n$ with rational coefficients.
- This finishes the proof. ■
- Given a graded module M, for any $d \in \mathbb{Z}$ we can define a new graded module by shifting the degrees:

$$M(d)_e = M_{d+e}.$$

Proof of Theorem

The proof is by induction on r, the number of variables.

- If $r = 0$, then M is simply a finite-dimensional graded vector space. $h_M(s) = 0$ for all large s.
- In the general case, take $K \subset M$ be the kernel of multiplication by x_r.
- We have the following exact sequence:

$$0 \to K(-1) \to M(-1) \xrightarrow{x_r} M \to M/x_rM \to 0.$$

- Taking the component of degree s of each term in the sequence and some careful linear algebra yields

$$h_M(s) - h_M(s-1) = h_{M/x_rM}(s) - h_K(s-1).$$

- Both K and M/x_rM are finitely generated modules over $k[x_1, \cdots, x_{r-1}]$.

Specific Case

- Given a projective variety $X \subset \mathbb{P}^n$, consider the graded module $S(X) = k[x_1, \cdots, x_r]/I(X)$, the coordinate ring of X.
- The grading on this module is also given by the degree.
- The Hilbert function corresponding to $S(X)$ is the Hilbert function of X.
- That is,

$$P_X(m) = h_X(m) = h_{S(X)}(m) = \dim(S(X)_m).$$

- Look at the vector space of all homogeneous polynomials of degree m on $\mathbb{P}^n$. The Hilbert function gives the codimension of the subspace of those vanishing on X.
- Now, given a curve X we can define the *genus* as

$$g = 1 - P_X(0).$$

Hilbert function: Example

Let $X = \{p_0, p_1, p_2\} \subset \mathbb{P}^2$.

- Then,

$$h_X(1) = \begin{cases} 2 & \text{if the points are collinear} \\ 3 & \text{if the points are not collinear} \end{cases}$$

- $h_X(1) = \dim(S(X)_1) = 3 - \dim(I(X)_1)$, where $I(X)_1$ is the space of degree 1 homogeneous polynomials vanishing at p_0, p_1, p_2.

Sources

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