Mutations of Polynomials

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Outline

Preliminary Definitions

- A lattice $L \cong \mathbb{Z}^2$ can be thought of as a coordinate grid
- An affine transformation $\varphi : L \to \mathbb{Z}$ is of the form $\varphi(v) = Av + b$
- The "linear part" of φ is $\varphi_o(v) = Av$
- Given a set of lattice points, their convex hull is the smallest convex polygon that contains all of the points.

The definitions on the following frames go back to Fomin and Zelevinsky in their work, "Cluster Algebras I, Foundations"

Definition of Mutation

- A mutation data is a pair (φ, h) where $\varphi : L \to \mathbb{Z}$ is a nonconstant affine transformation, and h is an element of the lattice and in the kernel of φ_o. h must also be of the form $1 + x^n y^m$ with $n, m \in \mathbb{Z}$
- Given a mutation datum (φ, h) and $f \in \mathbb{C}[N]$, write

$$f = \sum_{k \in \mathbb{Z}} f_k \qquad \text{where} \qquad f_k \in \mathbb{C}[(\varphi = k) \cap N].$$

- We say that f is (φ, h)-mutable if for all $k < 0$ we have that h^{-k} divides f_k.
- If f is (φ, h)-mutable, then the mutation of f, with respect to this mutation datum is the polynomial

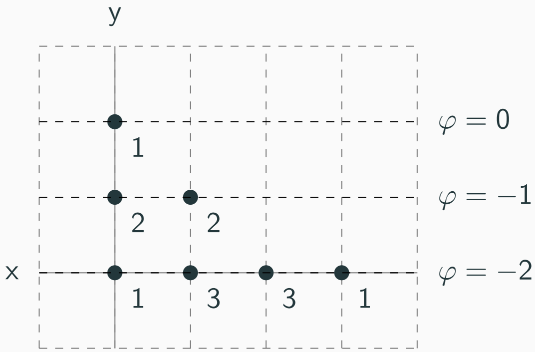
$$\mathrm{mut}_{(\varphi,h)} f = \sum_{k \in \mathbb{Z}} h^k f_k.$$

Definition of 0-mutable for Two variables

Let N be an affine lattice of rank 2. We define the set of 0-mutable polynomials on N in the following way:

- A monomial is 0-mutable
- The product $f = f_1 f_2$ is 0-mutable if and only if both factors, f_1, f_2 are 0-mutable.
- If f is 0-mutable, then every mutation of f is 0-mutable.

Example



This is the polynomial

$$1 + 3x + 3x^2 + x^3 + 2y + 2xy + y^2$$

Here we let $\varphi = y - 2$ (the height function minus two)

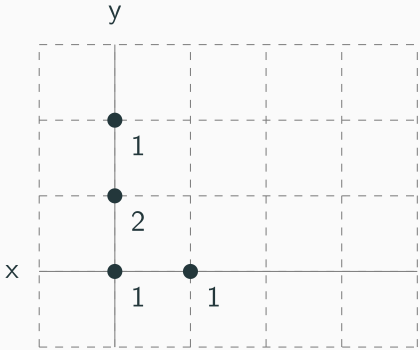
Thus, any polynomial in just x will be in the kernel of the linear part (x)

Choose $h = 1 + x$, and notice

$1+3x+3x^2+x^3 = (1+x)^3$

$2y + 2xy = 2y(1 + x)$

Example



This is the polynomial

$$1 + x + 2y + y^2$$

This is the polynomial we obtain from the previous mutation, and notice we can continue to reduce (possibly by choosing an affine function that has vertical level sets)

Code

Early in the summer, I began work on a code project to help us run over examples of mutations. The functionality includes

1. random mutations
2. convex hull illustration
3. reduction

Code continued...

The reduction deserves special mention, as the algorithm may be enlightening for future work. The following is performed for each side of the polynomial's convex hull.

1. Find the direction determined by the side
2. Group the terms in the polynomial by which parallel line with this given direction they line on
3. Factor all of the polynomials determined by these groups
4. Check if factors are shared in a way that would allow for a viable mutation

Note, we can find always find at least one mutation for each side, and thus actually infinitely many, but we only keep the "minimal" mutation

Code Continued...

Once reduction was working, we were close to finding an algorithm to efficiently check if a polynomial was 0-mutable. As long as the a 0-mutable polynomial never required a mutation that strictly increased the number of terms to mutate down to a monomial, we would be done.

Unfortunately, a long search found a counterexample, given by the equation

$$1 + 3x + 3x^2 + x^3 + y + 3xy + 4x^2y + x^3y + x^2y^2$$

Rigid Maximally Mutable Polynomials

During our project, Alessio Corti, Matej Philip, and Andrea Petracci published results in their paper "Mirror Symmetry and Smoothing Gorenstein Toric Affine 3-Folds" related to what we were working on. In it they define "rigid maximally mutable" polynomials. Let f be a polynomial and let S be a set of mutation data. We define

1. $\psi(f) = \{\text{Mutation data } s = (\phi, h) | \text{f is s-mutable}\}$
2. $L(S) = \{f | \forall s \in S, \text{f is s-mutable}\}$

A laurent polynomial such that $L(\psi(f)) = \{\lambda f | \lambda \in \mathbb{C}\}$ is called rigidly maximally mutable.

The paper used a very high level proof to show that 0-mutable and RMM polynomials are equivalent, but I found a simple combinatorial proof showing all 0-mutable polynomials are RMM

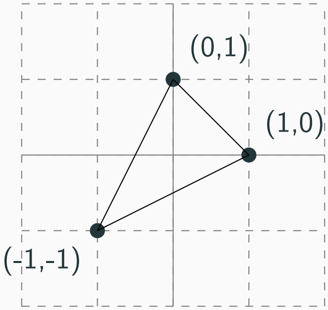
Some Noteworthy Results

- $\psi(f)$ for any polynomial is infinite, and will always carve out a well-defined convex hull in the plane
- reducible polynomials that are 0-mutable need not be rigid maximally mutable
- Rigid maximally mutable polynomials must have sides that are completely reducible

Mutating Polygons

Some Initial Definitions:

- Lattice N, its dual $M = \mathrm{Hom}(N, \mathbb{Z})$
- Fano Polygon: A convex lattice polygon such that the origin lies in the strict interior, and the vertices are primitive lattice vectors.

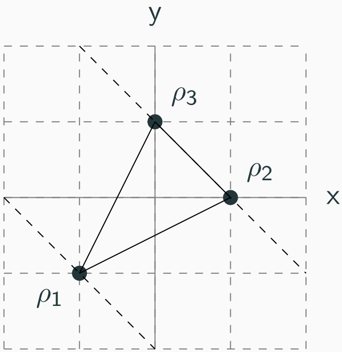


Combinatorial Mutation of Lattice Polygons

Akhtar et al., *Mirror Symmetry and the Classification of Orbifold del Pezzo Surfaces*

- Let $P \subset N_{\mathbb{Q}}$ be a lattice polygon. Choose an orientation of N.
- A *mutation data for P*, (h, f) is a choice of primitive vectors $h \in M$ and $f \in h^{\perp} \subset N$ satisfying the following conditions:
  - The vertices of P are labeled $\rho_1, \rho_2, \cdots$ counterclockwise, such that $h(\rho_1) = h_{max}$.
  - There is an edge $E_i = [\rho_i, \rho_{i+1}]$ such that $h(\rho_i) = h(\rho_{i+1}) = h_{min}$.
  - $\rho_{i+1} - \rho_i = wf$ where $w \geq -h_{min}$ is an integer.

Example



Consider the polygon $\mathrm{conv}((-1,-1),(1,0),(0,1))$ and the mutation data (h,f) where $h(x,y) = -x-y$ and $f = (-1,1)$.
Then, we have that $h_{\mathrm{max}} = h(\rho_1) = 2$, $h_{\mathrm{min}} = h(\rho_2) = h(\rho_3) = -1$, and $i = 2$.

Two Cases

Mutating P with respect to the mutation data (h, f):

- Case 1: P has m vertices, $\rho_1, \cdots, \rho_m$, and ρ_1 is the unique maximum for h on P.

$$\rho'_j = \begin{cases} \rho_j & 1 \leq j \leq i \\ \rho_j + h(\rho_j) f & i < j \leq m \\ \rho_1 + h_{max} f & j = m+1 \end{cases}$$

- Case 2: P has $m+1$ vertices, $\rho_1, \cdots, \rho_{m+1}$, and $h(\rho_1) = h(\rho_{m+1}) = h_{max}$.

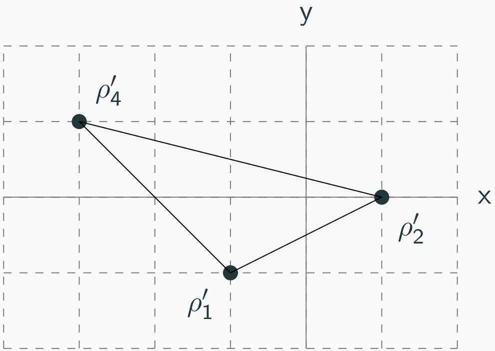
$$\rho'_j = \begin{cases} \rho_j & 1 \leq j \leq i \\ \rho_j + h(\rho_j) f & i < j \leq m \\ \rho_{m+1} + h_{max} f & j = m+1 \end{cases}$$

Example continued

Since we are in Case 1 the mutation is given by

$$\rho'_j = \begin{cases} \rho_j & 1 \leq j \leq 2 \\ \rho_j + h(\rho_j) f & j = 3 \\ \rho_1 + 2f & j = 4. \end{cases}$$

This yields the polygon on the right.



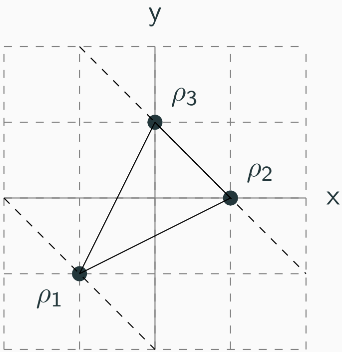
Another Definition

Akhtar, Coates, Galkin, Kasprzyk, *Minkowski Polynomials and Mutations*

- Let $P \subset N_{\mathbb{Q}}$ be a lattice polygon with vertices $\mathcal{V}(P)$, and let $w \in M$ be primitive.
- For each height $h \in \mathbb{Z}$, w defines a hyperplane $H_{w,h} := \{x \in N_{\mathbb{Q}} : w(x) = h\}$.
- Let $w_h(P) := \mathrm{conv}(H_{w,h} \cap P \cap N)$.
- The lattice polygon $F \subset N_{\mathbb{Q}}$ is a <u>factor</u> of P with respect to w if $w(F) = 0$, and if for every height $h_{\min} \leq h < 0$ there exists a lattice polygon $G_h \subset N_{\mathbb{Q}}$ satisfying

$$H_{w,h} \cap \mathcal{V}(P) \subseteq G_h + (-h)F \subseteq w_h(P).$$

Example Revisited



Consider the polygon $P = \text{conv}((-1,-1),(1,0),(0,1))$ and the mutation data (w, F) where $w(x,y) = -x - y$ and $F = \mathrm{conv}((0,0),(-1,1))$ is a factor of P with associated $G_{-1} = \{(1,0)\}$.

Definition of Combinatorial Mutation

- The combinatorial mutation given by width vector w, factor F, and polygons $\{G_h\}$ is the convex lattice polygon $\mathrm{mut}(P; w, F)$

$$= \mathrm{conv}\left(\bigcup_{h=h_{\min}^{-1} G_h \cup \bigcup_{h=0}^{h_{\max}(w_h(P)+hF)}} \right) \subset N_{\mathbb{Q}}$$

Example Revisited continued

Mutating P with respect to w and F yields

$$\begin{aligned}\mathrm{mut}(P; w, F) &= \mathrm{conv}\left(G_{-1} \cup \bigcup_{h=0,2} (w_h(P) + hF)\right)\\ &= \mathrm{conv}((1,0), (-1,-1), (-3,1)).\end{aligned}$$

which is the same polygon we got when mutating with the previous definition.

Results

- Given a Fano polygon, applying either mutation gives the same mutated polygon up to isomorphism.
- Thus, we can combine statements about the mutations from both mentioned papers.
- Let N be a 2-dimensional lattice and $g \in \mathbb{C}[x, y]$ a Laurent polynomial in two variables such that the convex hull of g is a Fano polygon P. Then, the set of the convex hulls up to isomorphism of all possible mutations of g is finite.

Future Work

If given the time, we would still like to explore the following questions

- is reduction monotone in some other variable, like number of sides of the convex hull?
- is there a simple combinatorial proof showing that any RMM polynomial is 0-mutable?
- what is the relationship with cluster algebras?

Acknowledgements

- We thank Professor Chris Woodward and Marco Castronovo for their mentorship
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