Venn diagrams in hypergraphs

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Introduction

Venn-Diagram

We say that hyper-graph contains k-Venn Diagram if there exist k sets A_1,A_2 $\dots A_k$. Such that each set $B_1 \cap B_2 \cap \dots B_k$ is non-empty, where B_i is either A_i or its complement for every i.

Our project

How many sets can our hypergraph have if it fails to contain a k-Venn diagram?

Background

This problem is in a way dual to a well known set-up in combinatorics.

VC-dimension

Family $\mathcal{F} \subset \mathcal{P}(n)$ shatters set $S \subset [n]$ if for all $A \subset S$ there exist $B \subset \mathcal{F}$ such that $B \cap S = A$. VC-dimension of $\mathcal{F}$ is then defined as

$$\mathrm{VC}(\mathcal{F}) = \max\{|S| : \mathcal{F} \text{ shatters } S\}$$

For VC($\mathcal{F}$)$= 2^k$ we get k-Venn diagram.

Sauer-Shelah Lemma

Sauer-Shelah [2].

For any set family $\mathcal{F} \subseteq 2^{[n]}$ we have

$$|\mathcal{F}| \leq \sum_{k=0}^{\mathrm{VC}(\mathcal{F})} \binom{n}{k}$$

and the bound is tight.

Example (Attaining the bound)

We can take all subsets of $[n]$ of size less than k. This set system shatters no set of size at least k. Obviously the number of such sets is exactly the bound.

Proof of Sauer-Shelah

Proof from [1].

- We prove a stronger version: $\mathcal{F}$ shatters at least $|\mathcal{F}|$ sets.
- We proceed by induction. Base case is trivial.
- Let $\mathcal{F}$ be a family of at least 2 sets. Fix $x \in \bigcup \mathcal{F}$ such that $\exists S \in \mathcal{F} : x \notin S$.
- Let $\mathcal{F}_1 = \{S \mid S \in \mathcal{F}, x \in S\}$ and $\mathcal{F}_2 = \mathcal{F} - \mathcal{F}_1$.
- Let $\mathcal{F}_1$ and $\mathcal{F}_2$ shatter $s_1 \geq |\mathcal{F}_1|$ and $s_2 \geq |\mathcal{F}_2|$ sets resp.
- Neither $\mathcal{F}_1$ nor $\mathcal{F}_2$ shatters a set containing x.
- If a set is shattered by $\mathcal{F}_1$ xor $\mathcal{F}_2$, it is also shattered by $\mathcal{F}$.
- S shattered by $\mathcal{F}_1$ and $\mathcal{F}_2 \Rightarrow S$ and $S \cup \{x\}$ shattered by $\mathcal{F}$.
- Thus $\mathcal{F}$ shatters at least $s_1 + s_2 \geq |\mathcal{F}_1| + |\mathcal{F}_2| = |\mathcal{F}|$.

$\square$

Our goals

Notation

We denote $f_k(n)$ as the maximum size of family $\mathcal{F}$ that does not form a k-Venn diagram.

Bounds [2]

$$f_2(n) = 4n - 2$$

$$f_3(n) = \Theta(n^3)$$

$$cn^{2^{k-1}-1} \leq f_k(n) \leq Cn^{2^k-1}$$

We believe the lower bound is tight. Our goal is to lower the order of the upper bound for $k = 4$ and for greater k.

References

[1] Sali Attila Anstee R.P., Rónyai Lajos. Shattering news. *Graphs and Combinatorics*, 18:59–73, March 2002.

[2] Peter Keevash, Imre Leader, Jason Long, and Adam Zsolt Wagner. The extremal number of venn diagrams, 2019.

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