Venn diagrams in hypergraphs

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Venn-Diagram

We say that hyper-graph contains $k$-Venn Diagram if there exist $k$ sets $A_1, A_2, \ldots, A_k$. Such that each set $B_1 \cap B_2 \cap \ldots B_k$ is non-empty, where $B_i$ is either $A_i$ or its complement for every $i$.

Our project

How many sets can our hypergraph have if it fails to contain a $k$-Venn diagram?
This problem is in a way dual to a well known set-up in combinatorics.

**VC-dimension**

Family $\mathcal{F} \subset \mathcal{P}(n)$ shatters set $S \subset [n]$ if for all $A \subset S$ there exist $B \subset \mathcal{F}$ such that $B \cap S = A$. VC-dimension of $\mathcal{F}$ is then defined as

$$VC(\mathcal{F}) = \max\{|S| : \mathcal{F} \text{ shatters } S\}$$

For $VC(\mathcal{F}) = 2^k$ we get $k$-Venn diagram.
Sauer-Shelah Lemma

For any set family $\mathcal{F} \subseteq 2^{[n]}$ we have

$$|\mathcal{F}| \leq \sum_{k=0}^{\text{VC}(\mathcal{F})} \binom{n}{k}$$

and the bound is tight.

Example (Attaining the bound)

We can take all subsets of $[n]$ of size less than $k$. This set system shatters no set of size at least $k$. Obviously the number of such sets is exactly the bound.
Proof of Sauer-Shelah

Proof from [1].

- We prove a stronger version: \( \mathcal{F} \) shatters at least \(|\mathcal{F}|\) sets.
- We proceed by induction. Base case is trivial.
- Let \( \mathcal{F} \) be a family of at least 2 sets. Fix \( x \in \bigcup \mathcal{F} \) such that \( \exists S \in \mathcal{F} : x \notin S \).
- Let \( \mathcal{F}_1 = \{ S | S \in \mathcal{F}, x \in S \} \) and \( \mathcal{F}_2 = \mathcal{F} - \mathcal{F}_1 \).
- Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) shatter \( s_1 \geq |\mathcal{F}_1| \) and \( s_2 \geq |\mathcal{F}_2| \) sets resp.
- Neither \( \mathcal{F}_1 \) nor \( \mathcal{F}_2 \) shatters a set containing \( x \).
- If a set is shattered by \( \mathcal{F}_1 \) xor \( \mathcal{F}_2 \), it is also shattered by \( \mathcal{F} \).
- \( S \) shattered by \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) \( \Rightarrow \) \( S \) and \( S \cup \{x\} \) shattered by \( \mathcal{F} \).
- Thus \( \mathcal{F} \) shatters at least \( s_1 + s_2 \geq |\mathcal{F}_1| + |\mathcal{F}_2| = |\mathcal{F}| \).
Our goals

**Notation**

We denote $f_k(n)$ as the maximum size of family $\mathcal{F}$ that does not form a $k$-Venn diagram.

**Bounds [2]**

\[
\begin{align*}
    f_2(n) &= 4n - 2 \\
    f_3(n) &= \Theta(n^3) \\
    cn^{2^{k-1}-1} &\leq f_k(n) \leq Cn^{2^{k-1}}
\end{align*}
\]

We believe the lower bound is tight. Our goal is to lower the order of the upper bound for $k = 4$ and for greater $k$. 
References


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