# A graph game from extremal combinatorics

Jan Soukup, Andrej Dedík
Supervised by: Sophie Spirkl

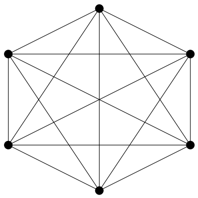
REU 2019, Rutgers University

This research is part of a project that has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 823748.

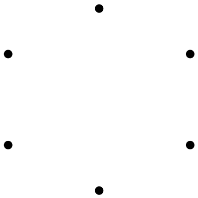
# Notation

- Graph $G :=$ Ordered pair $(V, E)$ of vertices and edges.
- $K_n :=$ Clique on $n$ vertices. I.e. $K_n = \left(V, \binom{V}{2}\right)$.
- Independent set in $G$ is a subset of vertices such that no two of them are connected by an edge.

Clique

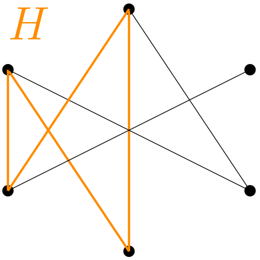


Independent set



# Notation

- $H$ is an induced subgraph of $G$ if there exists a subset $A$ of vertices of $G$, such that $G[A] = H$.



# The game

For a given integers $m$ and $k \geq 2$, and a fixed graph $H$ consider a following game played on a graph $G$

**Definition (Forcing game $\mathcal{F}(H, k, m)$)**

We start with a graph $G$ with $m$ vertices and no edges. Then players $A$ and $B$ take turns, as follows:

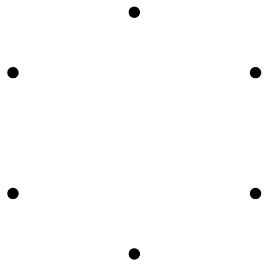
- Player $A$ either selects an independent set $S$ of size $k$ in $G$ or decides to stop the game.
- Player $B$ modifies $G$ by adding edges with both ends in $S$; he must add at least one edge, but may add more.

At the end of the game, Player $A$ wins if $G$ contains $H$ as an induced subgraph; and Player $B$ wins otherwise.

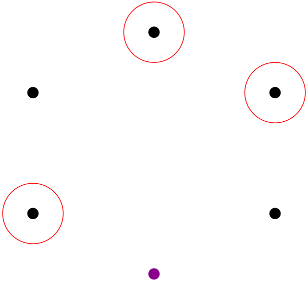
# Example

- $k = 3$
- $H = K_3$ (triangle)
- $m = 6$

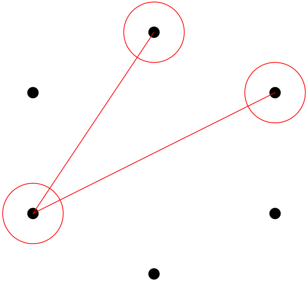
# Example. Turn 1.



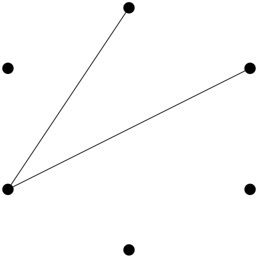
# Example. Turn 1: Player $A$.



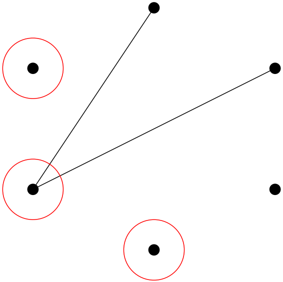
# Example. Turn 1: Player $B$.



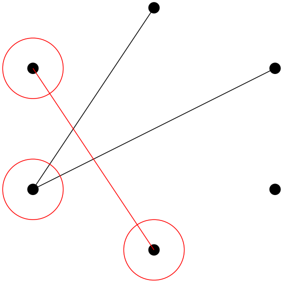
# Example. Turn 2.



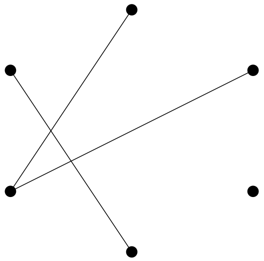
# Example. Turn 2. Player $A$.



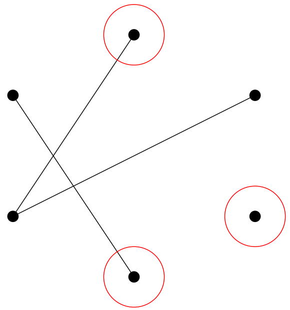
# Example. Turn 2. Player $B$.



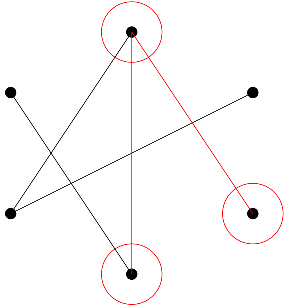
# Example. Turn 3.



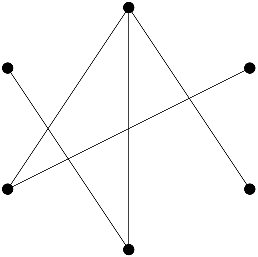
# Example. Turn 3. Player $A$.



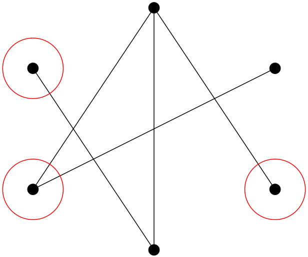
# Example. Turn 3. Player $B$.



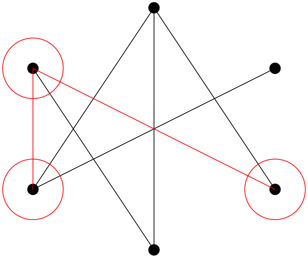
# Example. Turn 4.



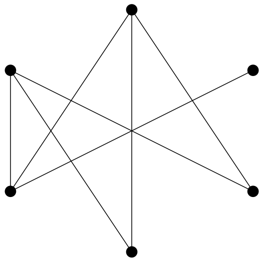
# Example. Turn 4. Player *A*.



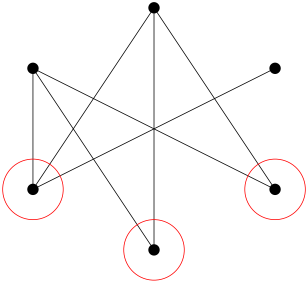
# Example. Turn 4. Player $B$.



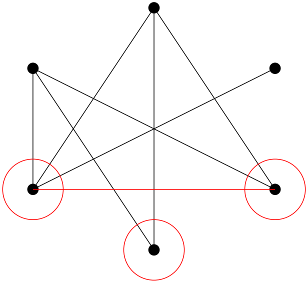
# Example. Turn 5.



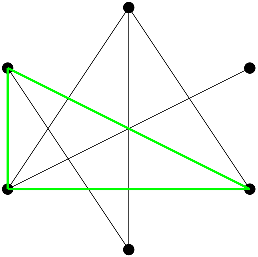
# Example. Turn 5. Player $A$.



# Example. Turn 5. Player $B$.



# Example. Player $A$ wins.



# Additional definitions

**Definition**

Let $N(H, k)$ be the minimal $m$ such that player $A$ can always win the game $\mathcal{F}(H, k, m)$. (No matter how the player $B$ plays).

**Definition (Ramsey number)**

Let $R(t, k)$ be the smallest $n$ such that every graph on at least $n$ vertices contains either $K_t$ or an independent set on $k$ vertices.

# Known facts

- For every $k$ and $H$, player $A$ wins for every sufficiently large graph. (i.e. $N(H,k) < \infty$).
- $N(H,k)$ is bounded from above by some function double exponential in $k$.
- $N(K_t,k) = R(t,k)$ (i.e. it coincides with the Ramsey number). And thus it is bounded from below by some function exponential in $k$.

# Our goal

- Find better bounds for $N(H, k)$.
- Explore other variants of the game. For instance when player $A$ does not see the graph $G$ and may chose to stop the game in his turn and:
  - Player $A$ loses when he choses a set which is not independent and he wins if $G$ contains induced $H$ in the end.
  - Player $A$ loses when he choses a set which is not independent and he wins if $G$ contains induced $H$ any time during the game.
  - Turn is skipped if player $A$ choses a set which is not independent and he wins if $G$ contains induced $H$ in the end.
  - Turn is skipped if player $A$ choses a set which is not independent and he wins if $G$ contains induced $H$ any time during the game.

For these variants it is not even known if player $A$ can always win on sufficiently large graphs.