THIS IS NOT THE LEGIT VERSION
DIMACS REU 2018
Project: Sphere Packings and Number Theory

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Apollonian circle packings

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Here is an illustrative example:

Crystallographic sphere packings

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This is a sphere packing that can be generated completely by a series of finite reflections. In a packing, the spheres densely fill space, and their interiors are disjoint.
In 2 dimensions, all circles are either disjoint or tangent – none of them intersect.

Crystallographic sphere packings

Here are some examples of crystallographic packings:

Inversive Coordinates

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$$v_s = \left(\frac{1}{\hat{r}}, \frac{1}{r}, \frac{x_1}{r}, \ldots \frac{x_n}{r} \right),$$

where $\frac{1}{\hat{r}}$ is the radius of sphere s after inversion through the unit sphere and $(x_1, \ldots, x_n)$ is the center of sphere s.

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where $\frac{1}{\hat{r}}$ is the radius of sphere s after inversion through the unit sphere and $(x_1, \dots, x_n)$ is the center of sphere s.
We call $\frac{1}{r}$ a "bend" b and $\frac{1}{\hat{r}}$ a "cobend" $\hat{b}$, so we usually see

$$v_s = \left(\hat{b}, b, bx_1, \dots bx_n \right).$$

Sphere Inversions

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- We apply sphere inversions to both spheres and planes, where planes are considered spheres of infinite radius.
- Spheres completely outside the mirror sphere will be inverted to spheres completely inside the mirror sphere, and vice versa.
- Sphere inversions preserve tangencies and angles.

Sphere Inversions

Here is an example in 2 dimensions:

Sphere Inversions

Crystallographic packings are generated by inversions of a *cluster* through its *cocluster* or *dual*.

- Circles in the cluster are either tangent to each other or disjoint
- Circles in the cocluster are either tangent, disjoint, or orthogonal to the circles in the cluster.

Sphere Inversions

For polyhedral packings, the cluster is the collection of vertices of the polyhedron, and the dual is the collection of faces of the polyhedron – In other words, the vertices of the dual polyhedron. We'll explain clusters and coclusters for Vinberg coordinates soon.

Hyperbolic geometries

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There is a surprising connection between sphere packings and non-Euclidean geometries. Euclidean geometry is characterized by Euclid's *parallel postulate*, which states that the angles formed by two lines intersecting on one side of a third line sum to be less than π radians.

Hyperbolic geometries

These geometries have several models which are each used as is necessary. n-dimensional hyperbolic space, $\mathbb{H}^n$, may be modeled using:

- The *upper sheet model* is characterized by all $x \in \mathbb{R}^{n+1} \cup \{\infty\}$ satisfying $xQx^T = -1, x_0 > 0$ for Q a $(n+1) \times (n+1)$ real matrix having exactly 1 negative eigenvalue and n positive eigenvalues.

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- The *Klein disk model* is the projection of the upper sheet onto the closed n-dimensional ball . This will be most familiar through the tesselations of M. C. Escher.
- The *upper half-space model* consisting of $x \in \mathbb{R}^{n+1} \cup \{\infty\}, x_0 > 0$.

Each has its own advantages. The upper half-space model in particular is relevant to Zack's work, for example.

Hyperbolic geometries

Inversions still work; in fact, in $\mathbb{H}^{n+1}$, they preserve angles, distances and volumes, and they keep points in side that upper half-space.
Inversions also have another useful property; $\hat{b}b - |bz|^2 = -1$. As $|bz|^2 = \sum_i (bz_i)^2$, this function $f(b, \hat{b}, bz)$ is a *quadratic form* on the set of inversive coordinates, i.e. a polynomial where each term is of degree two.

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A fact about inversive coordinates

Consider the circle $(\hat{b}, b, bz)$. We know that its diameter gets mapped to the diameter of its image under inversion. The new points on the boundary and the diameter are the images of points at distance $|z| - r$ and $|z| + r$, which get mapped to points at distance $\frac{1}{|z|-r}$ and $\frac{1}{|z|+r}$, respectively. Thus the new diameter is of length $\frac{1}{|z|-r} - \frac{1}{|z|+r} = \frac{2r}{|z|^2-r^2}$. This is also equal to $2\hat{r}$, so $\hat{r} = \frac{r}{|z|^2-r^2}$; rearranging gets $|z|^2 - r^2 = \frac{r}{\hat{r}} = -r\hat{b}$. Dividing by $r^2 = 1/b^2$ gives $|bz|^2 - 1 = \hat{b}b$, and so we find that $\hat{b}b - |bz|^2 = -1$, putting us under the 2-sheeted hyperboloid model of hyperbolic space.

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We view our circles as lying on the border of $\mathbb{H}^3$, where our circles form the boundary of geodosic hemispheres.

Our quadratic form Q

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In bend notation, this simplifies to

$$\hat{b}b - |bz|^2 = -1$$

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$\hat{b}b - |bz|^2 = -1$ can be expressed as $\begin{pmatrix} \hat{b} & b & bz \end{pmatrix} \cdot Q \cdot \begin{pmatrix} \hat{b} \\ b \\ bz \end{pmatrix}$,

where

$$Q = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & 0 & \\ 0 & 0 & -1 & 0 & \\ 0 & 0 & 0 & -1 & \\ \vdots & & & & -1 \end{pmatrix}$$

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adjusted according to the dimension.
This gives us a bilinear form $< x, y >_Q = xQy^T$, where $< v, v >_Q = -1$, putting us on the 2-sheeted hyperboloid $Q = -1$ in $\mathbb{H}^3$.

Gram Matrices and Coxeter Diagrams

When describing a packing, we want to know about the relationship between different spheres, specifically what angle each sphere makes with the others. This information allows us to build the packing (up to rescaling through inversions).

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A Gram matrix can be used to encode this information.

$$G_{i,j} = {v_i v_j}_Q = \begin{cases} -1, & v_i = v_j \\ 1, & v_i || v_j \\ 0, & v_i \perp v_j \\ cos(\theta), & \theta_{v_i, v_j} \\ cosh(d), & d = \text{hyperbolic distance}(v_i, v_j) \end{cases}$$

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If i, j intersect at an angle of $\frac{\pi}{n}$, we draw $n - 2$ lines between vertices i, j.

Structure Theorem: Kontorovich-Nakamura

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Moreover, *every* crystallographic packing is generated by reflections of the cluster through the cocluster.
Using the Coxeter diagram, a crystallographic packing exists if and only if there exists an isolated cluster – a vertex or collection of vertices that are disjoint, orthogonal, or parallel to all other vertices (which form the cocluster).
The packing arises by reflecting the circles in the cluster through the circles in the cocluster.

Structure Theorem: Kontorovich-Nakamura

Vinberg's Algorithm

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$\mathbb{H}^3$ can be represented by the action of $\mathbb{R}_+$ on positive-definite 2nd-order Hermitian matrices H_2^+, and certain transformations of elements in H_2 that preserve H_2^+ induce the orientation-preserving and -reversing motions of $\mathbb{H}^3$, thus making these elements and their action the group of isometries of $\mathbb{H}^3$.

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Bianchi groups are discrete subgroups of these elements, so they represent discrete groups of isometries of $\mathbb{H}^3$.

Devora studied the Bianchi groups that are generated by finite reflections, since these can produce circle packings.

Vinberg's Algorithm

Zack's work concerns the group of isometries on $\mathbb{H}^{n+1}$ with respect to the quadratic forms $Q_d = \begin{pmatrix} -d & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$.

In both instances, the groups do not come with obvious generators, so Vinberg's algorithm is necessary for determining the Coxeter diagram structure and hence whether a packing is obtainable.

Vinberg's Algorithm

The general idea of Vinberg's algorithm is that it takes a lattice with a quadratic form, and returns the generators of the automorphism group preserving the lattice.

Visually, we can view the lattice as a tiling of $\mathbb{H}^n$. Vinberg's algorithm finds the fundamental domain, which is a polyhedron. If the polyhedron is finite, the algorithm terminates after it has found all the generators. Otherwise, it runs infinitely.

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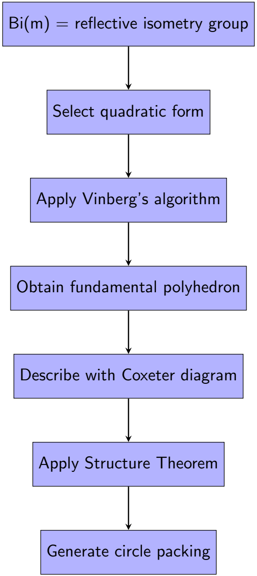
So how do Bianchi groups become circle packings? Using Vinberg's algorithm! (Much gratitude to Beloliptesky and McLeod)

From Bianchi groups to circle packings



From Bianchi groups to circle packings

Bi(1) yields the Apollonian strip packing with either isolated vertex due to symmetry

Integrality of Bianchi groups

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But how can we prove it?

Furthermore, if a packing is not integral, is there a way of proving that?

Integrality

Recall that crystallographic packings are generated by reflections of the cluster through the cocluster. We can represent these reflections as $V.R$, where V is a linearly independent matrix of the inversive coordinates of the cluster (and part of its orbit, if needed) and each R is a matrix corresponding to the mirror circle in the cocluster we are reflecting through.

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This formulation requires us to know EVERYTHING about the cluster circles to know about the bends of the packing, because R is right-acting.

But, if we could reformulate this to be a left-action, we could know about the bends of the packing just by knowing about the bends of the cluster! So, we are looking for Bend matrices B such that $B.V = V.R$.

Integrality

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Although many Bianchi packings do not initially have integral bends in V, most of them can be rescaled to integrality. And although many corresponding Bend matrices are not integral, they can be conjugated to integrality as well.

Devora is still working on properly conjugating all Bend matrices of integral packings to prove integrality.

Non-integrality

What about packings that we observe to be nonintegral?

Non-integrality

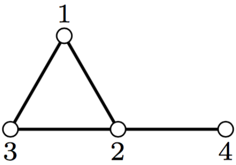
What about packings that we observe to be nonintegral?

If we can find a nonintegral relation between

Isolated clusters?

If you look closely, you'll see that not all sets of generators have isolated clusters.

Obtained by Vinberg's algorithm applied to $-x_0^2 + x_1^2 + x_2^2 + x_3^2$.



(Incorrectly) obtained from $\hat{B}i(3)$.

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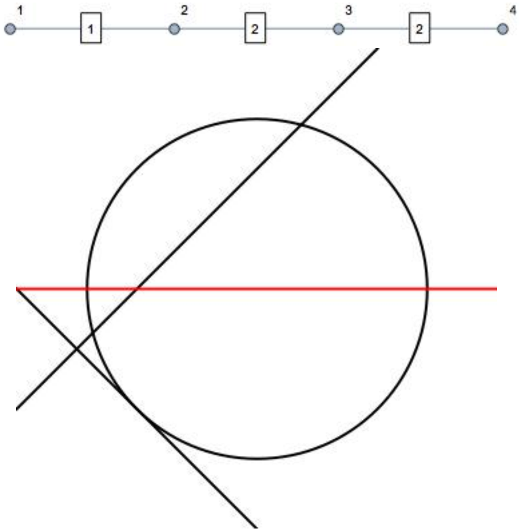
This entails choosing a circle in the space of the circles giving the Coxeter diagram and doubling the entire configuration about it.

Doubling example



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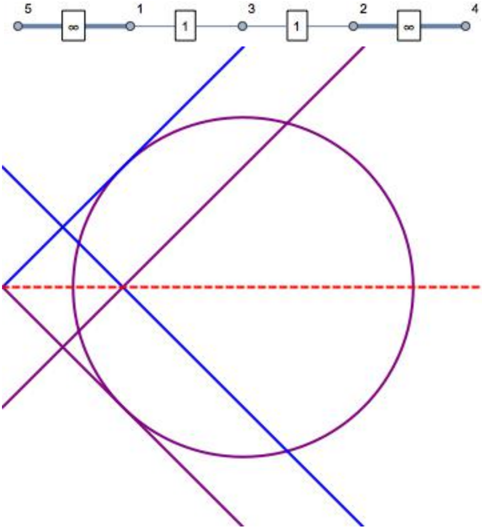

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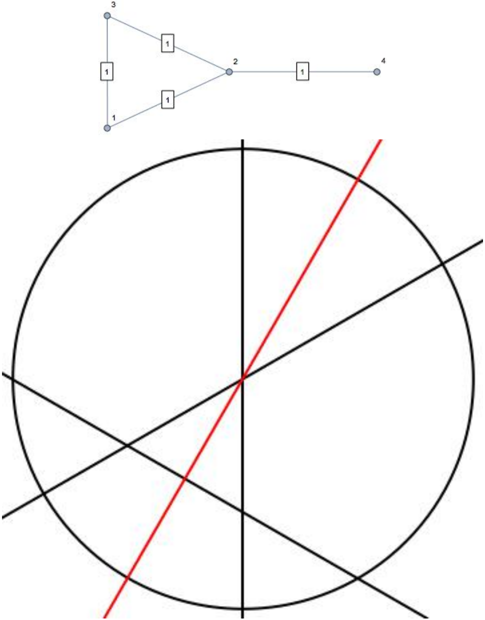
This entails choosing a circle in the space of the circles giving the Coxeter diagram and doubling the entire configuration about it.
Why? Consider the group of mirrors generated before and after doubling. The latter is a subgroup of the former, and hence this reveals to us structure that before was inaccessible.

Doubling in $\hat{B}i(3)$

One insight in Kontorovich & Nakamura's 2017 paper was the observation that the $\hat{B}i(3)$ Coxeter diagram did not represent the full group of mirrors:

Doubling in $\hat{Bi}(3)$



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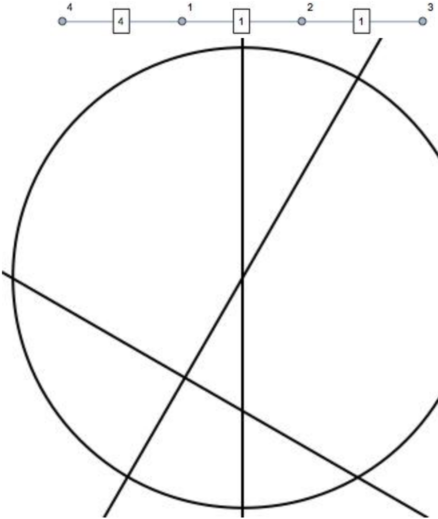

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Through a further series of operations, we can transform the diagram 

into the diagram .

Doubling in $\hat{B}i(3)$

Through a further series of operations, we can transform the diagram 

into the diagram . However, this was done less systematically; it primarily derived from looking amongst the orbit of the original generators acting on themselves until a new configuration was found, having tangencies and finite volume.

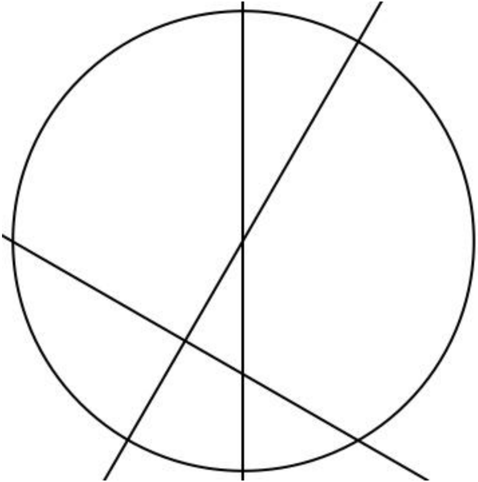
Volume?

What's this you say about volume? Aren't we looking at planar configurations?

Volume?

What's this you say about volume? Aren't we looking at planar configurations? Yes, but there is a deep connection between the configuration of circles in $\mathbb{R}^n$ and the upper half-space model of $\mathbb{H}^{n+1}$. Kontorovich and Nakamura proved that a crystallographic packing corresponds to a configuration of planes with interiors intersecting to form finite volume in $\mathbb{H}^{n+1}$.

Volume?



Volume

Vinberg proves an algorithmic approach to verifying whether a configuration has finite volume in this sense, which involves checking the Coxeter diagram's subdiagrams against the list of known irreducible groups.

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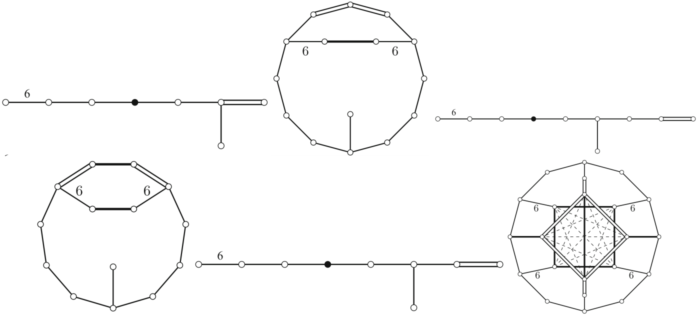

Resolving questions about existence of packings

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Resolving questions about existence of packings

What's something that all of these have in common? They all feature the diagram 

as a subdiagram!

Resolving questions about existence of packings

What's something that all of these have in common? They all feature the diagram  as a subdiagram! So, if we apply the known transformation for  into , and then a suitable action on the remainder of the nodes in the Coxeter diagram, then hopefully we will obtain a finite-volume diagram representing one such desired subgroup of mirrors.

Results

Polyhedral Packings

basics: interested in combinatorially distinct polyhedra, eg 3-connected planar graphs that are not isomorphic

Polyhedral Packings:Koebe-Andreev-Thurston

KAT gives us cluster + cocluster

Polyhedral Packings: Structure Theorem (K-N)

takes us from KAT to an infinite packing, from there we can look at bends to find integrality

Polyhedral Packings: Methods

plantri to get polyhedron data, then into code written in mathematica that spits out supercluster, packing, inversive coords, bend matrices,gram

Polyhedral Packings: Findings?

previously known integral polyhedra: tetrahedron, square pyramid, hex pyramid, gluings of
(define gluings)

Polyhedral Packings: Findings?

new integral polyhedron: 6v7f_2
new rational but not integral polyhedron: 8v9f_3
plus more? (7v8f_9 and 7v9f_8?)

Polyhedral Packings: Example proof

proof of nonintegrality of 8v9f_3?
alternatively, proof of integrality (which doesn't really exist yet) of 6v7f_2 which is probably more interesting

Polyhedral Packings: Website

pull up the website?? might be awkward

References

We are much indebted to the following papers:
Kontorovich-Nakamura paper Vinberg paper Beloliptesky-McLeod McLeod thesis Milnor, Vinberg (hyp. vol.) Belolipetsky survey

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