Apollonian circle packings

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What is an Apollonian circle packing? Here is an illustrative example:
Crystallographic sphere packings

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The Apollonian packing is a type of crystallographic sphere packing. This is a sphere packing that can be generated completely by a series of finite reflections. In a packing, the spheres densely fill space, and their interiors are disjoint. In 2 dimensions, all circles are either disjoint or tangent – none of them intersect.
Here are some examples of crystallographic packings:
Inversive Coordinates

Inversive coordinates are a convenient way of identifying spheres in a packing:

\[ v_s = \left( \frac{1}{r}, r, x_1 r, \ldots, x_n r \right), \]

where \( \frac{1}{r} \) is the radius of sphere \( s \) after inversion through the unit sphere and \( (x_1, \ldots, x_n) \) is the center of sphere \( s \). We call \( r \) a “bend” and \( \frac{1}{r} \) a “cobend” \( \hat{b} \), so we usually see \( v_s = (\hat{b}, b, \hat{b}x_1, \ldots, \hat{b}x_n) \).
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Sphere Inversions

To generate crystallographic packings, we use sphere inversions through a ”mirror sphere”. This sends points at a distance of d from the center of the mirror sphere to a distance of 1/d from the center of the mirror sphere.
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▶ Spheres completely outside the mirror sphere will be inverted to spheres completely inside the mirror sphere, and vice versa.
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- We apply sphere inversions to both spheres and planes, where planes are considered spheres of infinite radius.
- Spheres completely outside the mirror sphere will be inverted to spheres completely inside the mirror sphere, and vice versa.
- Sphere inversions preserve tangencies and angles.
Sphere Inversions

Here is an example in 2 dimensions:
Sphere Inversions

Crystallographic packings are generated by inversions of a cluster through its cocluster or dual.

- Circles in the cluster are either tangent to each other or disjoint
- Circles in the cocluster are either tangent, disjoint, or orthogonal to the circles in the cluster.
For polyhedral packings, the cluster is the collection of vertices of the polyhedron, and the dual is the collection of faces of the polyhedron – In other words, the vertices of the dual polyhedron. We’ll explain clusters and coclusters for Vinberg coordinates soon.
Hyperbolic geometries

There is a surprising connection between sphere packings and non-Euclidean geometries.
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There is a surprising connection between sphere packings and non-Euclidean geometries. Euclidean geometry is characterized by Euclid’s \textit{parallel postulate}, which states that the angles formed by two lines intersecting on one side of a third line sum to be less than $\pi$ radians.
Hyperbolic geometries

These geometries have several models which are each used as is necessary. $n$-dimensional hyperbolic space, $\mathbb{H}^n$, may be modeled using:

- The upper sheet model is characterized by all $x \in \mathbb{R}^{n+1} \cup \{\infty\}$ satisfying $xQx^T = -1$, $x_0 > 0$ for $Q$ a $(n + 1) \times (n + 1)$ real matrix having exactly 1 negative eigenvalue and $n$ positive eigenvalues.
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- The *Klein disk model* is the projection of the upper sheet onto the closed $n$-dimensional ball. This will be most familiar through the tesselations of M. C. Escher.

- The *upper half-space model* consisting of $x \in \mathbb{R}^{n+1} \cup \{\infty\}, x_0 > 0$.

Each has its own advantages. The upper half-space model in particular is relevant to Zack’s work, for example.
Inversions still work; in fact, in $\mathbb{H}^{n+1}$, they preserve angles, distances and volumes, and they keep points in side that upper half-space.

Inversions also have another useful property; $\hat{b}b - |bz|^2 = -1$. As $|bz|^2 = \sum_i (bz_i)^2$, this function $f(b, \hat{b}, bz)$ is a quadratic form on the set of inversive coordinates, i.e. a polynomial where each term is of degree two.
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A fact about inversive coordinates

Consider the circle \((\hat{b}, b, bz)\). We know that its diameter gets mapped to the diameter of its image under inversion. The new points on the boundary and the diameter are the images of points at distance \(|z| - r\) and \(|z| + r\), which get mapped to points at distance \(\frac{1}{|z| - r}\) and \(\frac{1}{|z| + r}\), respectively. Thus the new diameter is of length \(\frac{1}{|z| - r} - \frac{1}{|z| + r} = \frac{2r}{|z|^2 - r^2}\). This is also equal to \(2\hat{r}\), so \(\hat{r} = \frac{r}{|z|^2 - r^2}\); rearranging gets \(|z|^2 - r^2 = \frac{r}{\hat{r}} = -\hat{b}^2\). Dividing by \(r^2 = 1/b^2\) gives \(|bz|^2 - 1 = \hat{b}b\), and so we find that \(\hat{b}b - |bz|^2 = -1\), putting us under the 2-sheeted hyperboloid model of hyperbolic space.
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\[
\hat{r} = \frac{r}{|z|^2-r^2};
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We view our circles as lying on the border of \(H^3\), where our circles form the boundary of geodosic hemispheres.
Our quadratic form $Q$

The laws governing circle inversion allows us to easily find $\hat{b}$ in terms of $b$ and the circle’s center $z$. 

\[ \hat{r} = \frac{1}{2} \left( \frac{1}{|z|} - r - \frac{1}{|z|} + r \right) \]

In bend notation, this simplifies to

\[ \hat{b} - \frac{|b|}{|z|} = -1 \]
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$$\hat{b}b - |bz|^2 = -1$$

can be expressed as

$$(\hat{b} \ b \ bz) \cdot Q \cdot \begin{pmatrix} b \\ bz \end{pmatrix},$$

where

$$Q = \begin{pmatrix} 
0 & 1/2 & 0 & 0 & \cdots \\
1/2 & 0 & 0 & 0 & \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\vdots & & & & -1 \\
\end{pmatrix}$$

adjusted according to the dimension.
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\[ \hat{b}b - |bz|^2 = -1 \] can be expressed as \( (\hat{b} \ b \ bz) \cdot Q \cdot \begin{pmatrix} \hat{b} \\ b \\ bz \end{pmatrix} \),

where

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Q = \begin{pmatrix}
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\frac{1}{2} & 0 & 0 & 0 & & \\
0 & 0 & -1 & 0 & & \\
0 & 0 & 0 & -1 & & \\
& & & & & \ddots \\
& & & & & -1
\end{pmatrix}
\]

adjusted according to the dimension.

This gives us a bilinear form \(< x, y >_Q = x Q y^T\), where \(< v, v >_Q = -1\), putting us on the 2-sheeted hyperboloid \( Q = -1 \) in $\mathbb{H}^3$. 
Gram Matrices and Coxeter Diagrams

When describing a packing, we want to know about the relationship between different spheres, specifically what angle each sphere makes with the others. This information allows us to build the packing (up to rescaling through inversions).
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\[
G_{i,j} = v_i v_j Q = \begin{cases} 
-1, & v_i = v_j \\
1, & v_i \parallel v_j \\
0, & v_i \perp v_j \\
\cos(\theta), & \theta_{v_i,v_j} \\
\cosh(d), & d = \text{hyperbolic distance}(v_i, v_j)
\end{cases}
\]
A Coxeter diagram encodes the same information, minus the hyperbolic distance between two disjoint spheres.
A Coxeter diagram encodes the same information, minus the hyperbolic distance between two disjoint spheres. If $i, j$ intersect at an angle of $\frac{\pi}{n}$, we draw $n - 2$ lines between vertices $i, j$. 
Structure Theorem: Kontorovich-Nakamura

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Every orbit of the group generated by reflections of a cluster through its cocluster produces a crystallographic packing. Moreover, every crystallographic packing is generated by reflections of the cluster through the cocluster. Using the Coxeter diagram, a crystallographic packing exists if and only if there exists an isolated cluster – a vertex or collection of vertices that are disjoint, orthogonal, or parallel to all other vertices (which form the cocluster).
Structure Theorem: Kontorovich-Nakamura

Every orbit of the group generated by reflections of a cluster through its cocluster produces a crystallographic packing. Moreover, every crystallographic packing is generated by reflections of the cluster through the cocluster. Using the Coxeter diagram, a crystallographic packing exists if and only if there exists an isolated cluster – a vertex or collection of vertices that are disjoint, orthogonal, or parallel to all other vertices (which form the cocluster). The packing arises by reflecting the circles in the cluster through the circles in the cocluster.
Structure Theorem: Kontorovich-Nakamura
Vinberg’s Algorithm

Recall from the definition of the Coxeter diagram that it is produced with the generators of a group of mirrors. What if a given group of mirrors $\Gamma$ acting on $\mathbb{H}^{n+1}$ does not come with prespecified generators?

The data characterizing $P$ is equivalent to the generators of $\Gamma$!
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Devora’s work concerns the *Bianchi groups* 
\[ \hat{\mathcal{B}}(m) := \text{SL}_2(\mathbb{Q}[\sqrt{-m}]) \] where \( m \) is a square-free positive integer.
Vinberg’s Algorithm

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\[ \hat{\text{Bi}}(m) := SL_2(\mathbb{Q}[\sqrt{-m}]) \] where \( m \) is a square-free positive integer. \( \mathbb{H}^3 \) can be represented by the action of \( \mathbb{R}_+ \) on positive-definite 2nd-order Hermitian matrices \( H_2^+ \), and certain transformations of elements in \( H_2 \) that preserve \( H_2^+ \) induce the orientation-preserving and -reversing motions of \( \mathbb{H}^3 \), thus making these elements and their action the group of isometries of \( \mathbb{H}^3 \).
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Bianchi groups are discrete subgroups of these elements, so they represent discrete groups of isometries of \( \mathbb{H}^3 \).

Devora studied the Bianchi groups that are generated by finite reflections, since these can produce circle packings.
Zack’s work concerns the group of isometries on $\mathbb{H}^{n+1}$ with respect to the quadratic forms $Q_d = \begin{pmatrix} -d & & \\ & 1 & \\ & & \ddots & 1 \end{pmatrix}$.

In both instances, the groups do not come with obvious generators, so Vinberg’s algorithm is necessary for determining the Coxeter diagram structure and hence whether a packing is obtainable.
Vinberg’s Algorithm

The general idea of Vinberg’s algorithm is that it takes a lattice with a quadratic form, and returns the generators of the automorphism group preserving the lattice.

Visually, we can view the lattice as a tiling of $\mathbb{H}^n$. Vinberg’s algorithm finds the fundamental domain, which is a polyhedron. If the polyhedron is finite, the algorithm terminates after it has found all the generators. Otherwise, it runs infinitely.
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So how do Bianchi groups become circle packings? Using Vinberg’s algorithm! (Much gratitude to Beloliptesky and McLeod)
From Bianchi groups to circle packings

1. $\text{Bi}(m) = \text{reflective isometry group}$
2. Select quadratic form
3. Apply Vinberg’s algorithm
4. Obtain fundamental polyhedron
5. Describe with Coxeter diagram
6. Apply Structure Theorem
7. Generate circle packing
From Bianchi groups to circle packings

Bi(1) yields the Apollonian strip packing with either isolated vertex due to symmetry
Integrality of Bianchi groups

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But how can we prove it?
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But how can we prove it?

Furthermore, if a packing is not integral, is there a way of proving that?
Recall that crystallographic packings are generated by reflections of the cluster through the cocluster. We can represent these reflections as $V.R$, where $V$ is a linearly independent matrix of the inversive coordinates of the cluster (and part of its orbit, if needed) and each $R$ is a matrix corresponding to the mirror circle in the cocluster we are reflecting through.
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This formulation requires us to know EVERYTHING about the cluster circles to know about the bends of the packing, because $R$ is right-acting.

But, if we could reformulate this to be a left-action, we could know about the bends of the packing just by knowing about the bends of the cluster! So, we are looking for Bend matrices $B$ such that $B.V = V.R$. 

**Integrality**
If we have an integral $B$ matrix for every reflection in the cocluster, and $V$ has integral bends, then the bends of the packing will definitively be integral as well.
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Devora is still working on properly conjugating all Bend matrices of integral packings to prove integrality.
Non-integrality

What about packings that we observe to be nonintegral?
Non-integrality

What about packings that we observe to be nonintegral? If we can find a nonintegral relation between
Isolated clusters?

If you look closely, you’ll see that not all sets of generators have isolated clusters.

Obtained by Vinberg’s algorithm applied to $-x_0^2 + x_1^2 + x_2^2 + x_3^2$.

(Incorrectly) obtained from $\hat{Bi}(3)$. 
Isolated clusters?

What hope do we have to get a packing in these cases?
Doubling

What hope do we have to get a packing in these cases? We have at our disposal a technique known as *doubling*. 
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Doubling example
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Doubling

What hope do we have to get a packing in these cases? We have at our disposal a technique known as doubling. This entails choosing a circle in the space of the circles giving the Coxeter diagram and doubling the entire configuration about it. Why? Consider the group of mirrors generated before and after doubling. The latter is a subgroup of the former, and hence this reveals to us structure that before was inaccessible.
Doubling in $\hat{Bi}(3)$

One insight in Kontorovich & Nakamura’s 2017 paper was the observation that the $\hat{Bi}(3)$ Coxeter diagram did not represent the full group of mirrors:
Doubling in $\hat{Bi}(3)$
Doubling in $\hat{Bi}(3)$
Through a further series of operations, we can transform the diagram into the diagram.
Doubling in $\hat{Bi}(3)$

Through a further series of operations, we can transform the diagram $\bullet \quad 4 \quad \bullet \quad 1 \quad 2 \quad 1 \quad 3$ into the diagram $\bullet \quad 4 \quad \bullet \quad 1 \quad 2 \quad 1 \quad 3$. However, this was done less systematically; it primarily derived from looking amongst the orbit of the original generators acting on themselves until a new configuration was found, having tangencies and finite volume.
Volume?

What’s this you say about volume? Aren’t we looking at planar configurations?
What’s this you say about volume? Aren’t we looking at planar configurations? Yes, but there is a deep connection between the configuration of circles in $\mathbb{R}^n$ and the upper half-space model of $\mathbb{H}^{n+1}$. Kontorovich and Nakamura proved that a crystallographic packing corresponds to a configuration of planes with interiors intersecting to form finite volume in $\mathbb{H}^{n+1}$. 
Volume?
Vinberg proves an algorithmic approach to verifying whether a configuration has finite volume in this sense, which involves checking the Coxeter diagram’s subdiagrams against the list of known irreducible groups.
Resolving questions about existence of packings

With these tools under our belt, we are now able to start looking at questions of whether packings exist within certain configurations.
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With these tools under our belt, we are now able to start looking at questions of whether packings exist within certain configurations. These techniques have allowed us to attack the following diagrams that lack isolated clusters:
Resolving questions about existence of packings

What’s something that all of these have in common?
Resolving questions about existence of packings

What’s something that all of these have in common? They all feature the diagram \( \begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet \\
4 & 4 & 1 & 1 & 2 & 1 & 3
\end{array} \) as a subdiagram!
Resolving questions about existence of packings

What’s something that all of these have in common? They all feature the diagram $\square \square \square \square \square \square \square$ as a subdiagram! So, if we apply the known transformation for $\square \square \square \square \square \square \square$ into $\square \square \square \square \square \square \square$, and then a suitable action on the remainder of the nodes in the Coxeter diagram, then hopefully we will obtain a finite-volume diagram representing one such desired subgroup of mirrors.
Results
Polyhedral Packings

basics: interested in combinatorially distinct polyhedra, eg 3-connected planar graphs that are not isomorphic
KAT gives us cluster + cocluster
takes us from KAT to an infinite packing, from there we can look at bends to find integrality
Polyhedral Packings: Methods

plantri to get polyhedron data, then into code written in mathematica that spits out supercluster, packing, inversive coords, bend matrices, gram
Polyhedral Packings: Findings?

previously known integral polyhedra: tetrahedron, square pyramid, hex pyramid, gluings of
(define gluings)
new integral polyhedron: 6v7f_2
new rational but not integral polyhedron: 8v9f_3
plus more? (7v8f_9 and 7v9f_8?)
proof of nonintegrality of 8v9f_3?
alternatively, proof of integrality (which doesn’t really exist yet) of 6v7f_2 which is probably more interesting
pull up the website?? might be awkward
We are much indebted to the following papers:
Kontorovich-Nakamura paper Vinberg paper Beloliptesky-McLeod
McLeod thesis Milnor, Vinberg (hyp. vol.) Belolipetsky survey
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