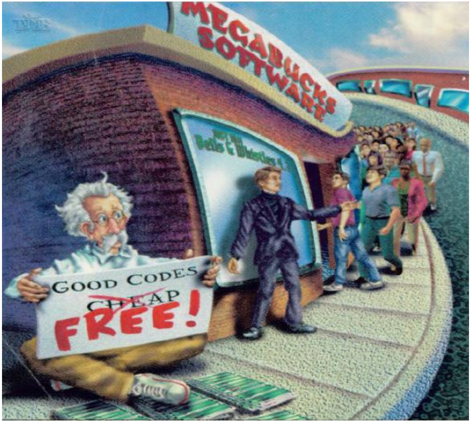
Service Rates of Codes and Vertex Covers of Graphs

Emina Soljanin, Rutgers



Joint work with

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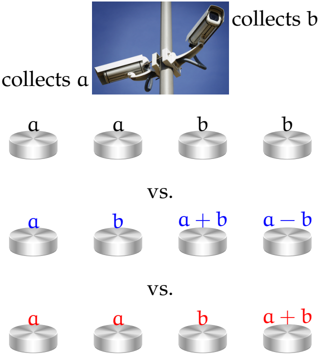
S. Kadhe, G. Matthews, C. Mayer, G. Yadgar

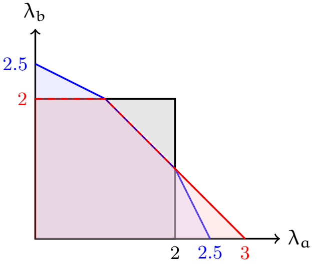
Demand for Low Latency

- Webpage download
  Amazon: 100ms $\sim$ costs 1% sales, Google: 1s $\sim$ page view drops 11%
- Interactive Tasks: 100ms - 150ms
- Online Gaming: 30ms
- Augmented Reality: 7ms - 20ms
- 5G, The Tactile Internet: 1ms

Service Rates of Codes

New applications create new performance metrics for codes.





Service Capacity Problem Formulation

System Model:

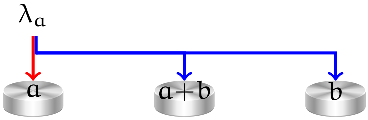
- k files $F_1, \ldots, F_k$ are stored redundantly across n servers.
- The size of each file and the size of data on each server are equal.
- Time to download a file from a server is exponential with rate 1.
- Requests to download F_i arrive at rate λ_i.

THE OBJECTIVE:

1. Determine the set of rates $(\lambda_1, \ldots, \lambda_k)$ that can be supported by the system implementing some common redundancy schemes.
2. Provide guidance on how to choose a redundancy scheme in order to maximize and/or shape the of region of supported arrival rates.

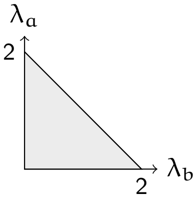
[3, 2] Simplex Code: $(a, b) \to (a, b, a+b)$

How can requests for file a be served?

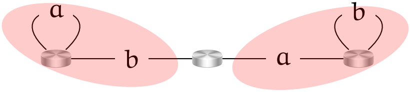


$\Rightarrow \lambda_a \leqslant 2$

Service Capacity Bound:

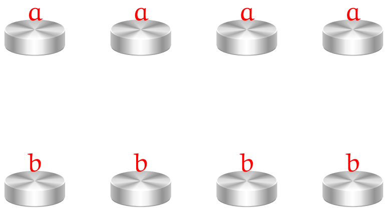


Relating to a graph covering problem:

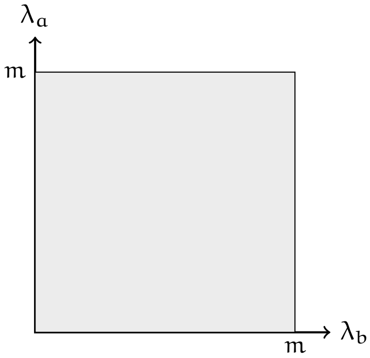


$\Rightarrow \lambda_a + \lambda_b \leqslant 2$

$[2m, 2]$ Repetition Code



Service Capacity Region:

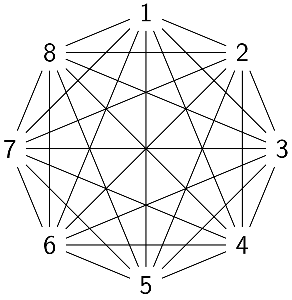


$[2m, 2]$ RS Code with no Systematic Nodes

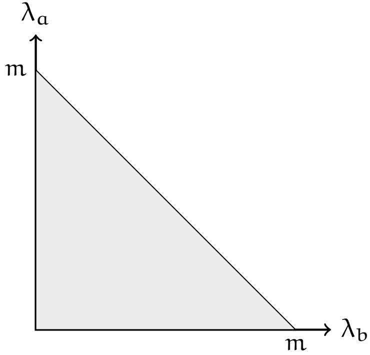
$a+b$ $a+\alpha b$ $a+\alpha^2 b$ $a+\alpha^3 b$ $a+\alpha^4 b$ $a+\alpha^5 b$ $a+\alpha^6 b$ $a+\alpha^7 b$

α is primitive in some F_q, $q \geqslant 8$

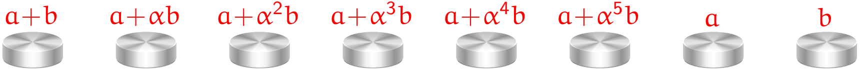
A graph to cover:



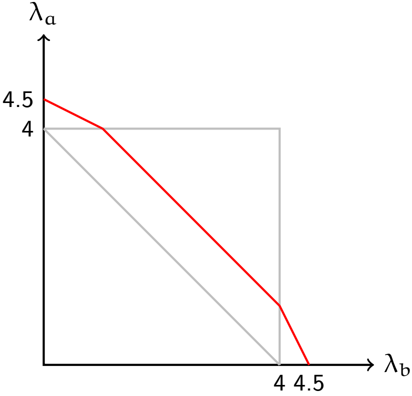
Service Capacity Region:



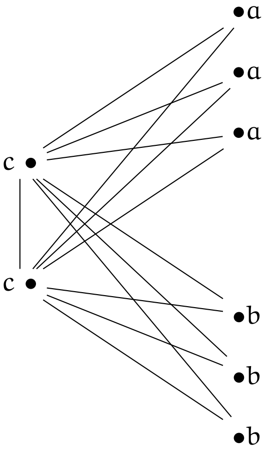
A Systematic [8, 2] RS Code



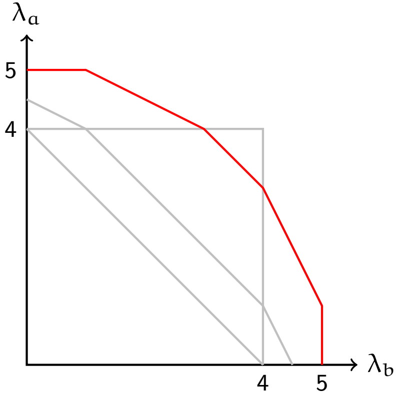
Service capacity region:



How About Both Coding and Replicating

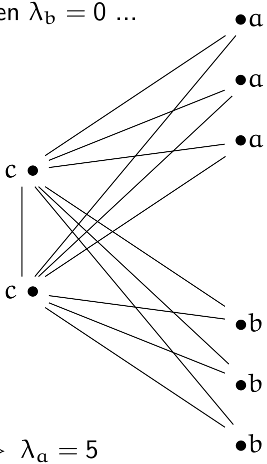


Service capacity region:



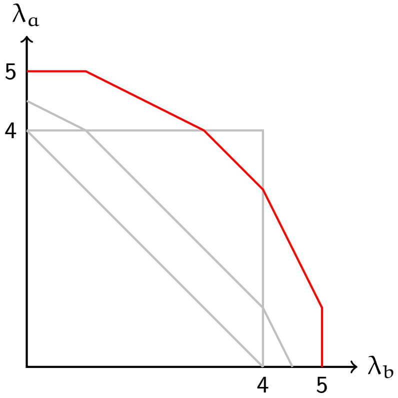
How About Both Coding and Replicating

When $\lambda_b = 0$ ...



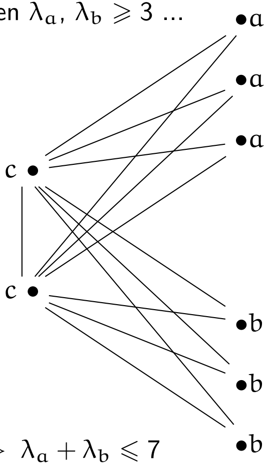
$\implies \lambda_a = 5$

Service capacity region:



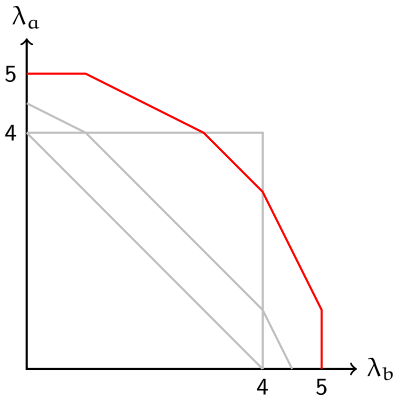
How About Both Coding and Replicating

When $\lambda_a, \lambda_b \geqslant 3$ ...



$\implies \lambda_a + \lambda_b \leqslant 7$

Service capacity region:



A General Theorem

Let there A systematic nodes for file a, B for file b, & C coded nodes, and assume that any 2 coded nodes or a coded and a systematic node can recover both a and b.

Then the service rate region is bounded by

$\lambda_a = 0$, $\lambda_b = 0$, $\lambda_a = \min\{(A+C)\mu,\ (A+\frac{B}{2}+\frac{C}{2})\mu\}$, and

$$L(\lambda_a) = \begin{cases} (B+C)\mu & \text{if } A > C \text{ and } 0 \leqslant \lambda_a \leqslant (A-C)\mu \\ -\frac{1}{2}\lambda_a + (\frac{A}{2}+B+\frac{C}{2})\mu & \text{if } A > C \text{ and } (A-C)\mu < \lambda_a \leqslant A\mu \\ -\frac{1}{2}\lambda_a + (\frac{A}{2}+B+\frac{C}{2})\mu & \text{if } A \leqslant C \text{ and } 0 \leqslant \lambda_a \leqslant A\mu \\ -\lambda_a + (A+B+\frac{C}{2})\mu & \text{if } A\mu < \lambda_a \leqslant (A+\frac{C}{2})\mu \\ -2\lambda_a + (2A+B+C)\mu & \text{if } B > C \text{ and } (A+\frac{C}{2})\mu < \lambda_a \leqslant A+C \\ -2\lambda_a + (2A+B+C)\mu & \text{if } B \leqslant C \text{ and } (A+\frac{C}{2})\mu < \lambda_a \leqslant (A+\frac{B}{2}+\frac{C}{2})\mu. \end{cases}$$

Dagstuhl development:

Gven a service region, find A, B, C s.t. $A+B+C$ is minimal.

Locally Repairable Codes (LRC) with Availability

A code has (r, t) availability if

- there are t disjoint repair groups for each data symbol &
- each repair group has at most r symbols.

A $(2, 3)$-availability code:

$$\{a, b, c\} \longrightarrow \{\boxed{a}, b, c, a+b, b+c, a+c, a+b+c\}$$

The minimum distance penalty for an $[n, k]$ code with (r, t) availability is

$$d_{\text{min}} \leqslant n - k + \left\lceil \frac{t(k-1)+1}{t(r-1)+1} \right\rceil + 2$$

There are code alphabet dependent bounds.

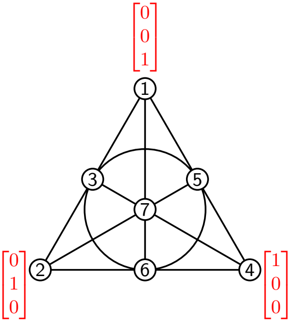
A Simplex Code Example

$$\{a, b, c\} \longrightarrow \{\boxed{a}, b, c, a+b, b+c, a+c, a+b+c\}$$

The $[7, 3, 4]$ Simplex code is a $(2, 3)$-availability code:

$$G_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Repair groups for position 1 are $\{2\&3, 4\&5, 6\&7\}$
(any two columns of G that sum to 1)



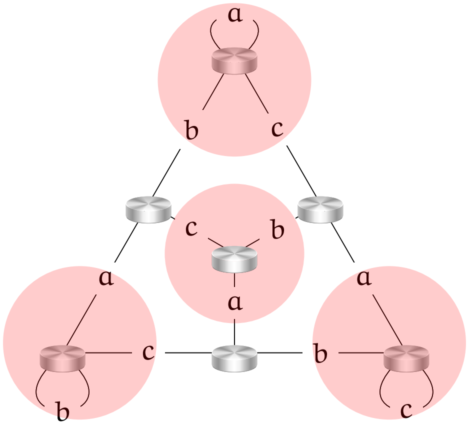
The Binary Simplex Code $\mathcal{S}_m$, $m \geqslant 2$

- G_m consist of all distinct nonzero vectors of $\mathbb{F}_2^m$
- $\mathcal{S}_m$ is a $[2^m - 1, m, 2^{m-1}]$ binary code.
- $\mathcal{S}_m$ has locality $r = 2$ and availability $t = 2^{m-1} - 1$.

Why do we care about these codes?

Considering rate, alphabet size, minimum distance, locality, availability, the binary simplex codes are in certain sense optimal.

[7, 3] Simplex Code



$[2^m - 1, m]$ Simplex Code

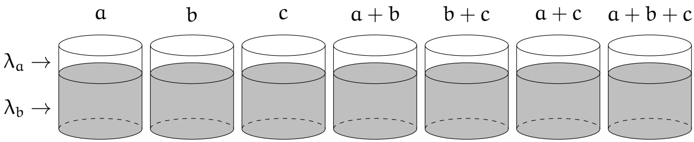
Theorem:

The service capacity region of the $[2^m - 1, m]$ Simplex coded system consists of all $\lambda_1, \ldots, \lambda_m$ s.t. $\lambda_1 + \cdots + \lambda_m \leqslant 2^{m-1}\mu$.

Proof Sketch for the Achevability:

Note that $\lambda_i \leqslant 2^{m-1}\mu$. Each server dedicates the fraction $\lambda_i/(2^{m-1}\mu)$ of its capacity solely to serving requests for file F_i.

$[7, 3]$ simplex code:



$[2^m - 1, m]$ Simplex Code

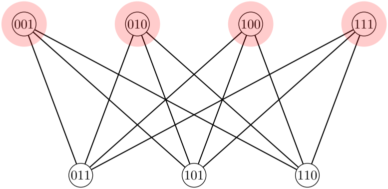
Theorem:

The service capacity region of the $[2^m - 1, m]$ Simplex coded system consists of all $\lambda_1, \ldots, \lambda_m$ s.t. $\lambda_1 + \cdots + \lambda_m \leqslant 2^{m-1}\mu$.

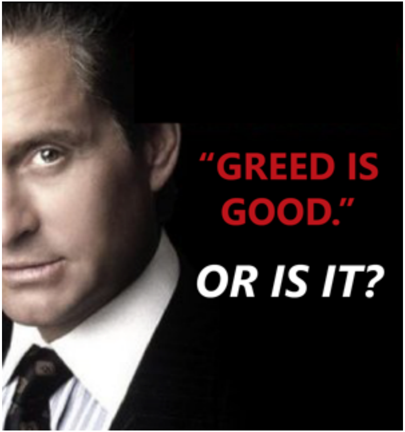
Proof Sketch for the Converse:

We consider graph Γ_m with $2^m - 1$ vertices labeled by all non-zero vectors of $\mathbb{F}_2^m$. Two vertices are connected iff their labels differ by 1. $\Longrightarrow$ Γ_m is a complete bipartite graph. The 2^{m-1} vertices with odd number of 1s cover each edge (recovery group) exactly once.

$[7, 3]$ simplex code:

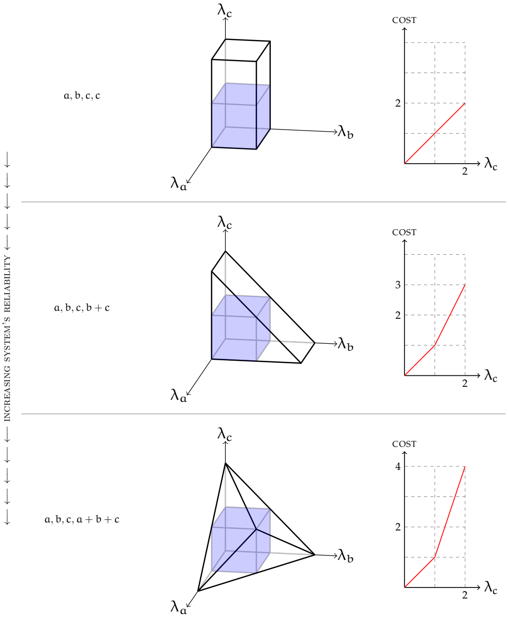


What about Systematic $[n, k]$ MDS Codes?



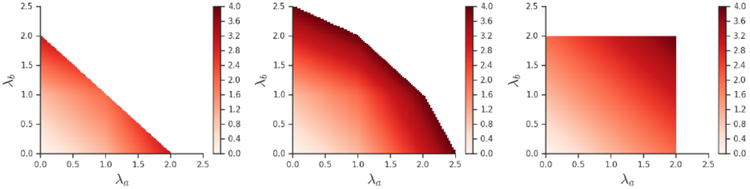
What about 1) water filling or 2) sharing?

Systems Issues



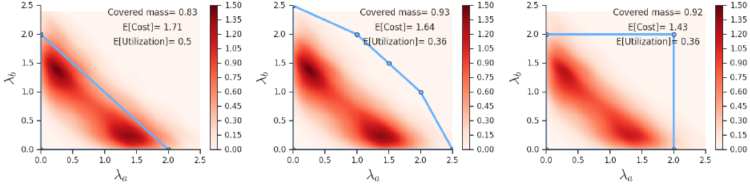
Systems Issues

Cost:



(a) $S(a, b, a+b)$ **(b)** $S(a, b, a+b, a+2b)$ **(c)** $S(a, a, b, b)$

Coverage and Utilization:



(a) $S(a, b, a+b)$ **(b)** $S(a, b, a+b, a+2b)$ **(c)** $S(a, a, b, b)$

and Σ robustness (related to Batch codes).