Spherical metrics with cone singularities

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Abstract: Given finitely many points $\{p_1, \ldots, p_n\}$ on a topological sphere S^2 and an n-tuple of positive real numbers $\{\beta_1, \ldots, \beta_n\}$, we will study the following problems which could be considered as a generalization of the classical uniformization theorem for spherical Riemann surfaces:

Problem 1: Does there exist a Riemannian metric g on $S^2 \setminus \{p_1, \ldots, p_n\}$ of constant Gauss curvature 1 that has cone singularities at p_i of cone angle $2\pi\beta_i$.

Problem 2: If there exist such metrics, how many different equivalent classes are there?

The conic condition in the first Problem means that locally near p_i, there exists a conformal coordinate z near p_i such that the metric g looks like

$$g \sim |z - p_i|^{2\beta_i - 2}|dz|^2.$$

These two problems have been solved in following special cases:

(i) All cones are ice-cream cones: $\beta_i \in (0, 1]$ for all i.

(ii) All but at most three of $\{\beta_i\}$ are integers.

(iii) The angle parameters are $(1/2, 1/2, 1/2, 3/2)$.

We now know the very non-trivial constraints on the angle parameters β_i. However in general (e.g. in case (iii)) the existence of such metrics depends on the position of the points (equivalently on the conformal class of the punctured Riemann surface) which is not well understood.

Concerning the second enumeration problem, we will study the following general open conjecture:

Conjecture 1. *With any prescribed points and prescribed cone angles, there are finitely many equivalence classes of spherical conic metrics.*

These problems have been studied by many mathematician using methods from different subjects: complex analysis, geometry, analytic theory of ODE, parabolic vector bundles on Riemann surfaces. The preliminary preparation for studying these problems are good knowledge of complex analysis and basics for Riemann surfaces.